The Laplace transform and discontinuous differential equations

Last class we defined the Laplace transform.

Definition. The *Laplace transform* of the function $y(t)$ is the function

$$Y(s) = \int_0^\infty y(t)\, e^{-st}\, dt.$$

This transform is an "operator" (a function on functions). It transforms the function $y(t)$ into the function $Y(s)$.

Notation: We often represent this operator using the script letter $\mathcal{L}$. In other words,

$$\mathcal{L}[y] = Y.$$

For example,

$$\mathcal{L}[1] = \frac{1}{s}\,,$$

$$\mathcal{L}[e^{at}] = \frac{1}{s-a}\,, \quad \text{and}$$

$$\mathcal{L}[\sin t] = \frac{1}{s^2+1}\,.$$

Note that even if $y(t)$ is defined for all t, the Laplace transform $Y(s)$ may not be defined for all s.

Properties of the Laplace transform There are two properties of the Laplace transform that make it well suited for solving linear differential equations:

1. $\mathcal{L}\left[\dfrac{dy}{dt}\right] = s\mathcal{L}[y] - y(0)$ ($\mathcal{L}$ turns differentiation into multiplication)
2. $\mathcal{L}$ is a linear transform:
   (a) $\mathcal{L}[y_1 + y_2] = \mathcal{L}[y_1] + \mathcal{L}[y_2]$
   (b) $\mathcal{L}[ky] = k\mathcal{L}[y]$ if k is a constant

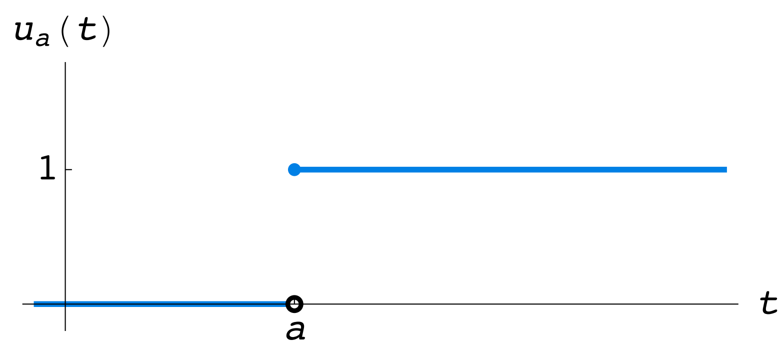
Discontinuous differential equations

The Laplace transform works well on linear differential equations that are discontinuous in one way or another.

Definition. The *Heaviside function* $u_a(t)$ is the function defined by

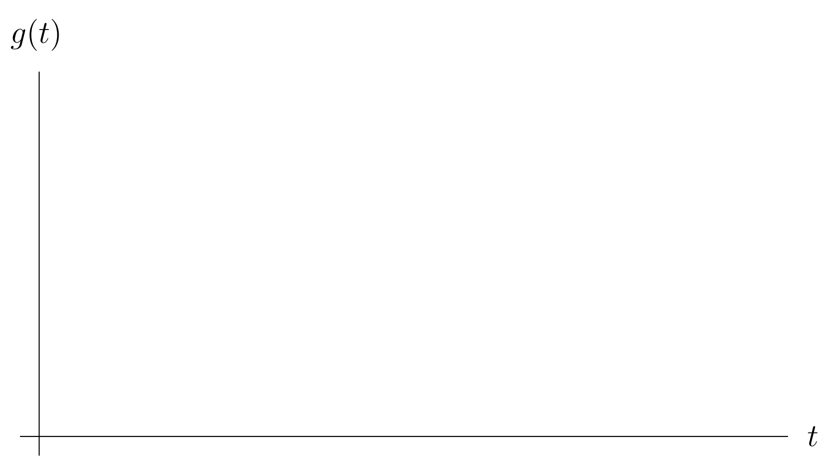
$$u_a(t) = \begin{cases} 0, & \text{if } t < a; \\ 1, & \text{if } t \geq a. \end{cases}$$

Thus $u_a(t)$ has a discontinuity at $t = a$ where it jumps from 0 to 1. Note that the `step(t)` function in `DETools` is the same function as $u_0(t)$ and that $u_a(t) =$ `step(t-a)`.



Here's how you can use the Heaviside function to avoid piecewise definitions:

Example. Consider $g(t) = 2t + u_1(t)(2 - 2t)$.



Laplace transforms are very convenient if we have discontinuous forcing. Remember the process for solving differential equations using Laplace transforms:

1. Transform both sides of the differential equation.
2. Determine $\mathcal{L}[y]$.
3. Compute the inverse Laplace transform of $\mathcal{L}[y]$.

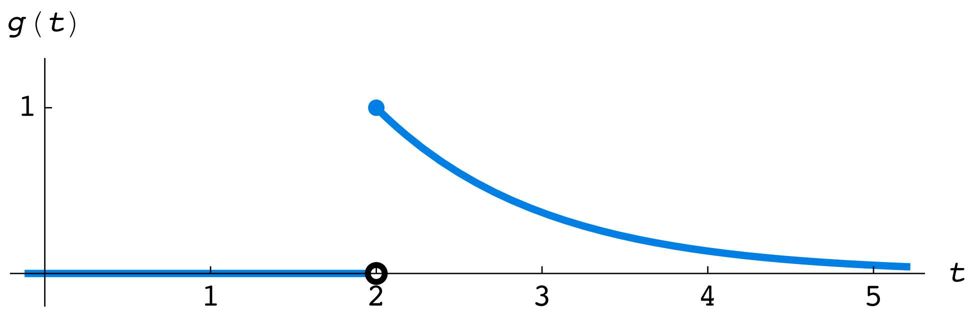
How do we calculate the Laplace transform of a discontinuous function?

Example. Let's calculate $\mathcal{L}[u_a]$ directly from the definition of $\mathcal{L}$.

In order to calculate inverse Laplace transforms, we need another property of the transform.

Rule 3: Shifting the t-axis. $\mathcal{L}[u_a(t)f(t-a)] = e^{-as}\mathcal{L}[f]$.

Example. Calculate $\mathcal{L}[g]$ where $g(t) = u_2(t)\, e^{-(t-2)}$.



Note: We usually use Rule 3 in reverse.

Why does the shifting rule work the way that it does?

Shifting the t-axis. Let's compute

$\mathcal{L}[u_a(t)f(t-a)] =$

Now let's see how we can use these properties of the Laplace transform to solve an initial-value problem that involves discontinuous forcing.

Example. Solve the initial-value problem

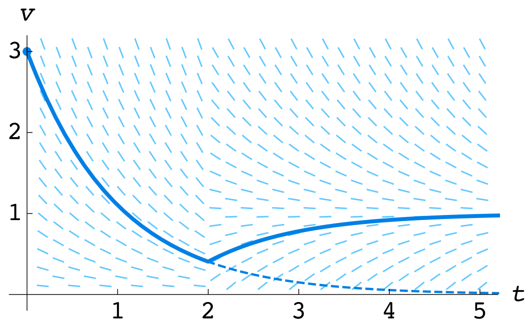
$$\frac{dv}{dt} + v = u_2(t), \quad v(0) = 3.$$

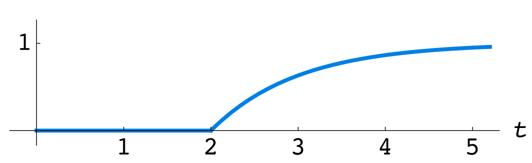
1. Transform both sides of the equation:

2. Solve for $\mathcal{L}[v]$:

3. Calculate the inverse Laplace transform:

Now let's plot the solution to the initial-value problem using `HPGSolver`. The graph of the solution is shown on the left below. The graph on the right is the graph of the function $u_2(t)(1 - e^{-(t-2)})$.





Laplace transforms and second-order equations

So far we have only applied the Laplace transform to first-order equations. Now we consider second-order equations.

Recall the rule for Laplace transforms of derivatives: $\mathcal{L}\left[\frac{dy}{dt}\right] = s\mathcal{L}[y] - y(0)$. What does this rule say about $\mathcal{L}\left[\frac{d^2y}{dt^2}\right]$?

Now that we have this rule, we also need to add to our table of Laplace transforms. Since sine and cosine often appear as parts of the solutions to second-order equations, let's determine their Laplace transforms.

There are a number of ways to compute these transforms—using integration by parts, using Euler's formula, and even using the fact that sine and cosine are solutions to certain very special second-order equations. *Mathematica* tells us that

$$\mathcal{L}[e^{i\omega t}] = \frac{1}{s - i\omega}.$$

Let's use this fact to determine $\mathcal{L}[\sin \omega t]$ and $\mathcal{L}[\cos \omega t]$.

Now that we know the transforms of sine and cosine, let's see how we use them.

Example. Compute

$$\mathcal{L}^{-1}\left[\frac{2s+1}{s^2+9}\right].$$

Now for a little practice with the third rule for transforms:

Example. Compute

$$\mathcal{L}^{-1}\left[\frac{8e^{-10s}}{(s^2+9)(s^2+1)}\right].$$