Quick recap of the end of last class

Consider the second-order, linear equation

$$a\,\frac{d^2y}{dt^2} + b\,\frac{dy}{dt} + cy = 0.$$

We have two ways to solve it:

1. By guessing $y(t) = e^{\lambda t}$. That technique was described in class on March 17 (see Section 2.3 in the text). Using that method we obtain the characteristic equation

$$a\lambda^2 + b\lambda + c = 0.$$

2. We can convert the second-order equation to a first-order system and use the eigenvalue/eigenvector approach. The characteristic equation for the eigenvalues of the associated system is

$$\lambda^2 + \frac{b}{a}\lambda + \frac{c}{a} = 0.$$

Useful observation: If λ is an eigenvalue, the vector $\mathbf{Y}_0 = \begin{pmatrix} 1 \\ \lambda \end{pmatrix}$ is *always* an associated eigenvector.

Note that these two characteristic equations are equivalent.

The observant student will note that we only discussed the case where the roots of the characteristic equation are real in Section 2.3 (see the top of page 187). Now we want to deal with all possibilities using what we know about complex eigenvalues and repeated eigenvalues.

All second-order linear equations

Let's use what we know about linear systems along with the eigenvector calculation to obtain solutions for all second-order linear equations. There are three cases:

1. Two real, distinct, nonzero eigenvalues λ_1 and λ_2:

2. A complex-conjugate pair of eigenvalues $\lambda = \alpha \pm i\beta$, with $\beta \neq 0$:

3. One nonzero real eigenvalue λ of multiplicity two:

Conclusion: We can determine the general solution of a homogeneous linear second-order equation

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0$$

immediately from the characteristic equation $a\lambda^2 + b\lambda + c = 0$.

YOU DO NOT NEED TO CALCULATE THE EIGENVECTORS OR EVEN REDUCE TO A FIRST-ORDER SYSTEM if you simply want to produce the general solution of a linear second-order equation.

Example. Let's compute the general solution to $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + y = 0$.

Application: We can apply what we have learned to the (damped) harmonic oscillator

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0.$$

In this case, we are assuming that the parameters m and k are positive and that $b \geq 0$. The characteristic equation $m\lambda^2 + b\lambda + k = 0$ has eigenvalues

$$\frac{-b \pm \sqrt{b^2 - 4mk}}{2m}.$$

There are four cases. In the first case, $b = 0$. This is the undamped case (see exercise 20 in Section 2.1).

1. $b = 0$:

The remaining three cases assume that $b > 0$, and they are based on the value of the discriminant $b^2 - 4mk$.

2. $b^2 - 4mk < 0$:

3. $b^2 - 4mk = 0$:

4. $b^2 - 4mk > 0$:

Example. Consider the one-parameter family of equations

$$\frac{d^2y}{dt^2} + b\frac{dy}{dt} + y = 0.$$

In this case, the characteristic equation is $\lambda^2 + b\lambda + 1 = 0$, and consequently, the eigenvalues are

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4}}{2}.$$

The value $b = 2$ is the critical value for this family.

We can see the progression from undamped to underdamped, to critically damped, and finally to overdamped with a Quicktime animation I have posted on the web site.

Summary of Phase Portraits

Assume $\det \mathbf{A} \neq 0$. Then zero is not an eigenvalue of $\mathbf{A}$.

1. Real and distinct eigenvalues
   (a) sink
   (b) saddle
   (c) source
2. Complex eigenvalues
   (a) spiral sink
   (b) center
   (c) spiral source
3. Real and repeated eigenvalues
   (a) sink with one eigenline in the phase portrait
   (b) source with one eigenline in the phase portrait
   (c) sink where every solution is a straight-line solution
   (d) source where every solution is a straight-line solution

What if $\det \mathbf{A} = 0$?

You can turn on the trace-determinant plane in the `LinearPhasePortraits` tool.

Forced equations

For the last four weeks of the semester, all of our differential equations have been autonomous. Now we turn to second-order equations that model systems that are subject to some type of external forcing. Here are three examples:

Example. The nonlinear pendulum with a pivot point that is subject to vertical oscillations. The motion of such a pendulum is governed by the second-order nonlinear equation

$$m\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + k\sin\theta = F\sin\theta\cos\omega t$$

where ω determines the frequency of the oscillations of the pivot point and F determines the amplitude of the oscillations. The `Pendulums` tool in `DETools` illustrates this system.

Example. The linear mass-spring system where the spring is subject to vertical oscillations. To model this system, we use the standard mass-spring system and add a term that corresponds to the force added to the system by the oscillations. We get

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = F\cos\omega t.$$

The `ForcedMassSpring` tool in `DETools` illustrates this system.

Example. The classic RLC circuit is also modeled by a linear, forced second-order equation. In `DETools`, it is modeled by an equation that involves both charge and current. In our text, we tend to use the equation

$$LC\frac{d^2v_c}{dt^2} + RC\frac{dv_c}{dt} + v_c = V_s(t)$$

where v_c is the voltage across the capacitor and R, L, and C are the resistance, inductance, and capacitance parameters. The forcing term $V_s(t)$ is a voltage source which can change with time. The `RLCCircuits` tool in `DETools` illustrates this system with a sinusoidal forcing function.

In class we will discuss forced linear equations only, but your second project will involve some experimentation with the forced pendulum.

Our success studying unforced linear systems was due in large part to the Linearity Principle. For forced linear equations, we are fortunate to have the Extended Linearity Principle.

Extended Linearity Principle Consider a nonhomogeneous equation (a forced equation)

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = g(t)$$

and its associated homogeneous equation (the unforced equation)

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0.$$

1. Suppose $y_p(t)$ is a particular solution of the nonhomogeneous equation and $y_h(t)$ is a solution of the associated homogeneous equation. Then $y_h(t) + y_p(t)$ is also a solution of the nonhomogeneous equation.

2. Suppose $y_p(t)$ and $y_q(t)$ are two solutions of the nonhomogeneous equation. Then $y_p(t) - y_q(t)$ is a solution of the associated homogeneous equation.

Therefore, if $k_1y_1(t)+k_2y_2(t)$ is the general solution of the associated homogeneous equation, then

$$k_1y_1(t) + k_2y_2(t) + y_p(t)$$

is the general solution of the nonhomogeneous equation.

This principle provides the basic framework that we will use to solve linear second-order forced equations. (At this point in the course, you should go back and review the method described in Section 1.8 for solving nonhomogeneous first-order linear equations.)

We already know how to find the general solution to the associated homogeneous equation, so we need only find one solution to the original equation.

Example 1. Consider the equation

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = e^{3t}.$$

Here's another example that looks similar but goes somewhat differently.

Example 2. Consider the equation

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = e^{-t}.$$