Flows for Vector Fields

In class on March 7, we discussed the flow $\varphi$ that is associated to a vector field. Given a vector field $V$ on $\mathbb{R}^n$, then the associated flow is a function

$$\varphi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$$

that is determined by the two conditions

$$\frac{\partial}{\partial t} \varphi(x, t) = V(\varphi(x, t)) \quad \text{and} \quad \varphi(x, 0) = x.$$

If you think of $V(x)$ as a velocity field, then the first condition says that the velocity vector of the path $\varphi(x, t)$ with $x$ fixed and $t$ varying agrees with the vector in the vector field. The second condition is an initial condition. It says that the path $\varphi(x, t)$ goes through $x$ at $t = 0$. The flow line through $x$ is the curve determined by this path. Flow lines are sometimes called streamlines or integral curves of the vector field.

Since $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, it is convenient to write $\varphi(x, t)$ as $\varphi_t(x)$. You should think of $\varphi_t(x)$ as the point you would arrive at if you started at $x$ and “flowed” for time $t$ according to the velocity field $V$.

**Example.** In class on March 7, we considered the gradient vector field

$$V(x, y) = -\frac{x}{2} \mathbf{i} + \frac{y}{2} \mathbf{j}.$$

This vector field is the gradient of $f : \mathbb{R}^2 \to \mathbb{R}$ where

$$f(x, y) = \frac{1}{4}(y^2 - x^2).$$

In class we derived the equation

$$\varphi_t(x, y) = \left( x e^{-t/2}, y e^{t/2} \right),$$

and we discussed why the flow lines are typically hyperbolic (pieces of hyperbolas) for this particular vector field.

Given the interpretation of $\varphi_t(x)$ given above, it makes sense to wonder if applying $\varphi_s$ after $\varphi_t$ is the same as applying $\varphi_{s+t}$. In fact, this is a basic property of flows, i.e.,

$$\varphi_{s+t} = \varphi_s \circ \varphi_t$$

for all $s, t \in \mathbb{R}$.  

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Exercise 1. Show that $\varphi_{s+t} = \varphi_s \circ \varphi_t$ holds for the flow

$$\varphi_t(x, y) = \left(x e^{-t/2}, ye^{t/2}\right).$$

Exercise 2. Another good example to keep in mind is the vector field

$$\mathbf{V}(x, y) = -yi + xj.$$ 

You should sketch some vectors in the vector field so that you get an idea of what this vector field looks like. Note that this vector field is not a gradient vector field.

If we write the flow for this vector field as $\varphi_t(x, y) = (x(t), y(t))$, then the two scalar functions $x(t)$ and $y(t)$ must satisfy the two equations

$$x'(t) = -y(t) \text{ and } y'(t) = x(t).$$

In other words, we want a pair of functions $x(t)$ and $y(t)$ such that the derivative of the first is the negative of the second and the derivative of the second is the first.

(a) Suppose that $k_1$ and $k_2$ are arbitrary constants (real numbers). Show that the two functions

$$x(t) = k_1 \cos t + k_2 \sin t \text{ and } y(t) = -k_2 \cos t + k_1 \sin t$$

satisfy the conditions given above for the flow associated to $\mathbf{V}$.

(b) Show that $\varphi_t(x, y) = (x \cos t - y \sin t, y \cos t + x \sin t)$.

(c) Verify that $\varphi_{s+t} = \varphi_s \circ \varphi_t$ for this flow.

(d) Using the formula for $\varphi_t(x, y)$, show that the flow lines for this flow are concentric circles centered at the origin.