1. (20 points) Consider the following 8 first-order equations:

1. $\frac{dy}{dt} = y^2 - 2y$ 2. $\frac{dy}{dt} = y^2 + 2y$ 3. $\frac{dy}{dt} = y^3 - 2y^2$ 4. $\frac{dy}{dt} = t^2 - 2t$

5. $\frac{dy}{dt} = 2t - ty$ 6. $\frac{dy}{dt} = 2t^2 - t^2 y$ 7. $\frac{dy}{dt} = 2ty - ty^2$ 8. $\frac{dy}{dt} = t^2 + 2t$

Four of the associated slope fields are shown on the next page. Pair the slope fields with their associated equations. Provide a brief justification for your choice. **You will not receive any credit unless you justify your selection.**

(a) The equation for slope field A is 3. My reason for choosing this answer is:

Field A is autonomous $\Rightarrow$ #1, 2, 3

Eq. solns $y=0$ and $y=2$ $\Rightarrow$ #1, 3

$\frac{dy}{dt} \leq 0$ for $y<2$ $\Rightarrow$ #3

(b) The equation for slope field B is 4. My reason for choosing this answer is:

$\frac{dy}{dt} = f(t)$ $\Rightarrow$ #4 or #8

$\frac{dy}{dt} = 0$ if $t=2$ $\Rightarrow$ #4

(c) The equation for slope field C is 7. My reason for choosing this answer is:

$\frac{dy}{dt} = f(t,y)$ $\Rightarrow$ #5, 6, 7

two eq. solutions $\Rightarrow$ #7

(d) The equation for slope field D is 5. My reason for choosing this answer is:

$\frac{dy}{dt} = f(t,y)$ $\Rightarrow$ #5, 6, 7

One eq. soln $\Rightarrow$ #5, 6

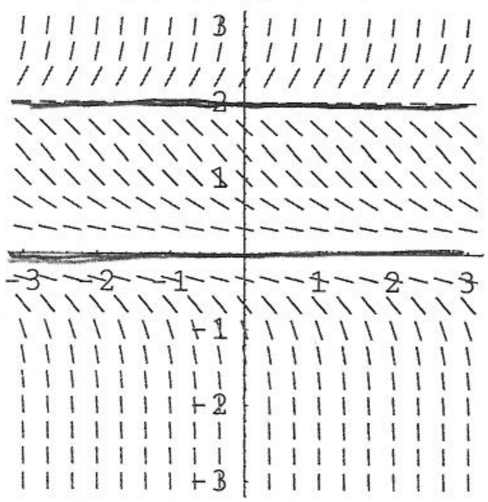
$\frac{dy}{dt}$ both pos and neg for $y<2$ $\Rightarrow$ #5

1. (continued) **Answer this question on the previous page.** The equations are provided here for your convenience:

1. $\dfrac{dy}{dt} = y^2 - 2y$
2. $\dfrac{dy}{dt} = y^2 + 2y$
3. $\dfrac{dy}{dt} = y^3 - 2y^2$
4. $\dfrac{dy}{dt} = t^2 - 2t$
5. $\dfrac{dy}{dt} = 2t - ty$
6. $\dfrac{dy}{dt} = 2t^2 - t^2y$
7. $\dfrac{dy}{dt} = 2ty - ty^2$
8. $\dfrac{dy}{dt} = t^2 + 2t$

autonomous

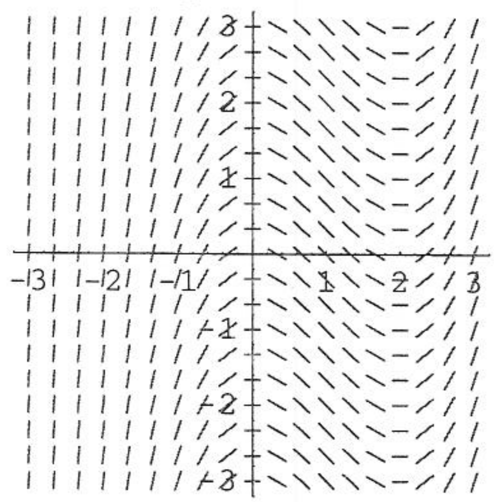
Slope Field A



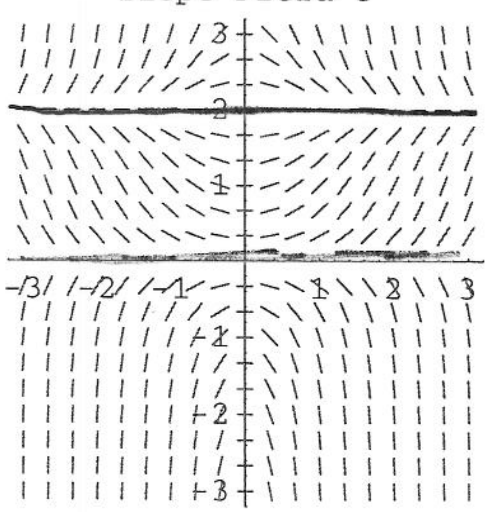
eq. solns

$\frac{dy}{dt} = f(t)$

Slope Field B

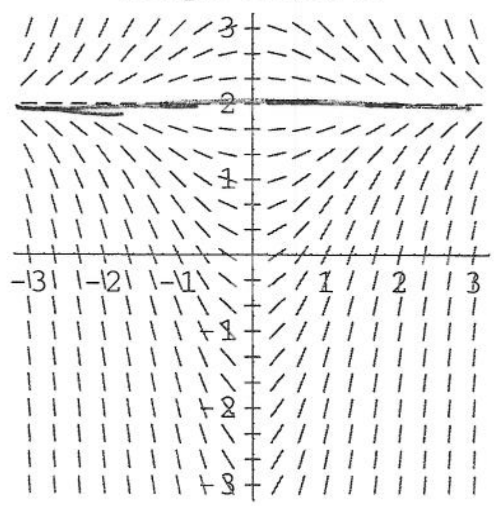


Slope Field C



eq. solns

Slope Field D



eq. soln

field depends on both t and y

2. (16 points)

(a) Calculate the first three Picard iterates $y_0(t)$, $y_1(t)$, and $y_2(t)$ for the initial-value problem

$$\frac{dy}{dt} = y^2 - 2t, \quad y(1) = 2.$$

$y_0(t) = 2$ for all t

$$y_1(t) = 2 + \int_1^t (4 - 2s)\, ds$$

$$= 2 + \left[4s - s^2\right]_1^t = 4t - t^2 + 2 - 4 + 1$$

$$= 4t - t^2 - 1$$

$$y_2(t) = 2 + \int_1^t (4s - s^2 - 1)^2 - 2s\, ds$$

$$(4s - s^2 - 1)^2 = (4s - s^2 - 1)(4s - s^2 - 1)$$

$$= 16s^2 - 4s^3 - 4s - 4s^3 + s^4 + s^2$$

$$- 4s + s^2 + 1$$

$$= s^4 - 8s^3 + 18s^2 - 8s + 1$$

$$y_2(t) = 2 + \int_1^t (s^4 - 8s^3 + 18s^2 - 10s + 1)\, ds$$

$$= 2 + \left[\frac{s^5}{5} - 2s^4 + 6s^3 - 5s^2 + s\right]_1^t$$

$$= \frac{t^5}{5} - 2t^4 + 6t^3 - 5t^2 + t +$$

$$\left(-\frac{1}{5} + 2 - 6 + 5 - 1\right) + 2$$

$$= \frac{t^5}{5} - 2t^4 + 6t^3 - 5t^2 + t + \frac{9}{5}$$

2. (continued)

(b) In one paragraph, explain how the initial-value problem

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

can be rewritten as an integral equation. (Be precise about the theorems that are used in this derivation.) Then explain how the integral equation leads to the iteration involved in Picard iteration.

Using the Fundamental Theorem of Calculus, we can write the solution as

$$y(t) - y(t_0) = \int_{t_0}^{t} y'(s)\, ds$$

$$= \int_{t_0}^{t} f(s, y(s))\, ds$$

$$\Rightarrow \quad y(t) = y(t_0) + \int_{t_0}^{t} f(s, y(s))\, ds.$$

This integral equation is satisfied by the solution to the initial-value problem. Unfortunately, the function y appears on both sides. Picard iteration replaces this equation with the iterative procedure

$$y_0(t) = y_0 \quad \text{for all } t$$

$$y_{k+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_k(s))\, ds.$$

3. (16 points) Consider the differential equation

$$\frac{dy}{dt} = \left(\frac{t^3}{1+t^4}\right) y + \frac{2}{1+t^4}.$$

(a) Show that $y(t) = 2t$ is a solution to this equation.

LHS: $\frac{dy}{dt} = 2$

RHS: $\left(\frac{t^3}{1+t^4}\right)(2t) + \frac{2}{1+t^4} = \frac{2(t^4+1)}{1+t^4} = 2$

$\Rightarrow y(t) = 2t$ is a solution.

(b) Find the general solution of this equation.

To find the general solution, we need to solve the associated homogeneous equation: $\frac{dy}{dt} = \left(\frac{t^3}{1+t^4}\right) y$.

We know that $y(t) = k e^{\int \frac{t^3}{1+t^4} dt}$

$\int \frac{t^3}{1+t^4} dt$ has $u = 1+t^4 \Rightarrow du = 4t^3$

$\int \frac{t^3}{1+t^4} dt = \frac{1}{4} \int \frac{1}{u} du = \frac{1}{4} \ln u = \frac{1}{4} \ln(1+t^4)$

$= \ln\left(\sqrt[4]{1+t^4}\right)$

$k e^{\int \frac{t^3}{1+t^4} dt} = k \sqrt[4]{1+t^4}$

Gen soln of NH: $y(t) = 2t + k\sqrt[4]{1+t^4}$ where k is an arbitrary constant

4. (16 points) Consider the initial-value problem

$$\frac{dy}{dt} = \frac{1}{2y+4}, \quad y(0) = -3.$$

(a) Find a formula for the solution.

Separate variables:

$$\int (2y+4)\,dy = \int 1\,dt$$

$$y^2 + 4y = t + c$$

$y(0) = -3 \Rightarrow 9 - 12 = c \Rightarrow c = -3$

We get $y^2 + 4y + (3-t) = 0$

$$y = \frac{-4 \pm \sqrt{16 - 4(3-t)}}{2} = -2 \pm \sqrt{1+t}$$

To satisfy $y(0) = -3 \Rightarrow y = -2 - \sqrt{1+t}$

(b) What is the domain of definition of this solution?

We must have $1+t > 0 \Rightarrow t > -1$

(c) What happens to the solution as it approaches the limits of its domain of definition? Why can't the solution be extended for more time?

As $t \to -1$ from above, $y(t) \to -2$

Since $\frac{dy}{dt}$ is undefined at $y = -2$, we cannot extend the solution across the line $y = -2$.

5. (16 points) Consider the initial-value problem

$$\frac{dy}{dt} = y^2 - 2y + 1, \quad y(0) = 2.$$

(a) Calculate the results of Euler's method applied to this initial-value problem on the interval $[0, 2]$ with 4 subdivisions. (Make sure that you show enough calculations so that we can see that you know the method.) Then graph your results. Make sure that you label the axes on your graph and clearly indicate the scale on each axis. You can use a calculator and do all calculations to 2 decimal places if you wish.

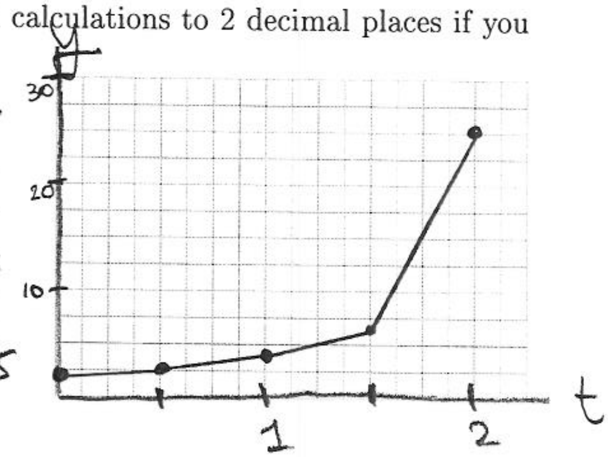
$\Delta t = \frac{2}{4} = .5$

$y_0 = 2$

$y_{k+1} = y_k + f(t_k, y_k)\,\Delta t$

$f(t_k, y_k) = y_k^2 - 2y_k + 1$

t_k	y_k
0	2
.5	2.5
1	3.62
1.5	7.05
2	25.35



(b) Solve this initial-value problem by separating variables and comment upon your results in part (a).

$$\frac{dy}{dt} = (y-1)^2 \Rightarrow \int (y-1)^{-2}\,dy = \int 1\,dt$$

$$\frac{1}{1-y} = t + c$$

$$y(0) = 2 \Rightarrow c = -1$$

$$\frac{1}{1-y} = t - 1 \Rightarrow 1 - y = \frac{1}{t-1} \Rightarrow y = 1 - \frac{1}{t-1}$$

Note that $\lim_{t\to 1^-} y = \lim_{t \to 1^-} \frac{t-2}{t-1} = \frac{t-2}{t-1}$

$= \infty$

So the actual solution blows up in finite time (at $t = 1$). The Euler approximation is misleading

6. (16 points) Locate the bifurcation value(s) for the one-parameter family of differential equations

$$\frac{dy}{dt} = (y^2 - \alpha)(y^2 - 2)$$

and draw phase lines for values of the parameter below, above, at, and between each bifurcation value.

We have eq. points at $\pm\sqrt{2}$ for all α.

We have eq. points at $\pm\sqrt{\alpha}$ for $\alpha \geq 0$.

bifurcation values : $\alpha = 0$ and $\alpha = 2$