1. (16 points) Solve the initial-value problem

$$\frac{dx}{dt} = 4x - 2y$$

$$\frac{dy}{dt} = x + y$$

$$A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$$

with $(x(0), y(0)) = (-1, -2)$.

$$\det(A - \lambda I) = \det\begin{pmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{pmatrix} = (\lambda-4)(\lambda-1)+2$$
$$= \lambda^2 - 5\lambda + 6$$
$$= (\lambda-3)(\lambda-2)$$

$\lambda = 3$ evecs: $\begin{cases} 4x_0 - 2y_0 = 3x_0 \\ x_0 + y_0 = 3y_0 \end{cases} \Rightarrow x_0 = 2y_0$

$\lambda = 2$ evecs: $\begin{cases} 4x_0 - 2y_0 = 2x_0 \\ x_0 + y_0 = 2y_0 \end{cases} \Rightarrow x_0 = y_0$

Gen solution:

$$Y(t) = k_1 e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + k_2 e^{3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

IVP: $Y(0) = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$

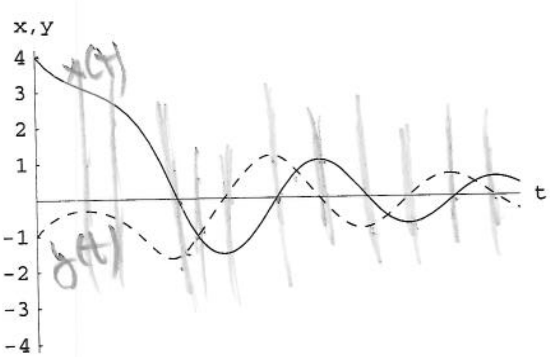
$$\Rightarrow \begin{pmatrix} -1 \\ -2 \end{pmatrix} = \begin{pmatrix} k_1 + 2k_2 \\ k_1 + k_2 \end{pmatrix}$$

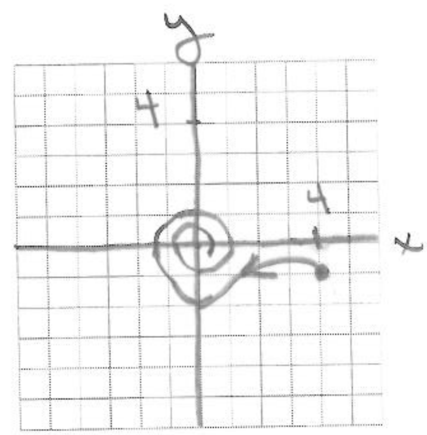
$\Rightarrow k_2 = 1$ and $k_1 = -3$

$$Y(t) = -3e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2e^{3t} - 3e^{2t} \\ e^{3t} - 3e^{2t} \end{pmatrix}$$

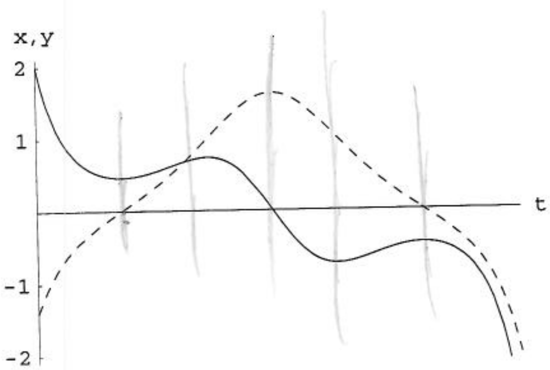
2. (16 points) In each part of this problem, $x(t)$- and $y(t)$-graphs for a solution are shown. The $x(t)$-graph is the solid curve, and the $y(t)$-graph is the dashed curve. Using the graph paper on the right, sketch the solution curve corresponding to these graphs and indicate the direction the solution goes at t increases by placing at least one arrowhead on your curve. Make sure that the axes in your drawing are clearly labeled with a variable and a scale. Sketch only that part of the solution curve that corresponds to the interval of t-values shown in the graphs.

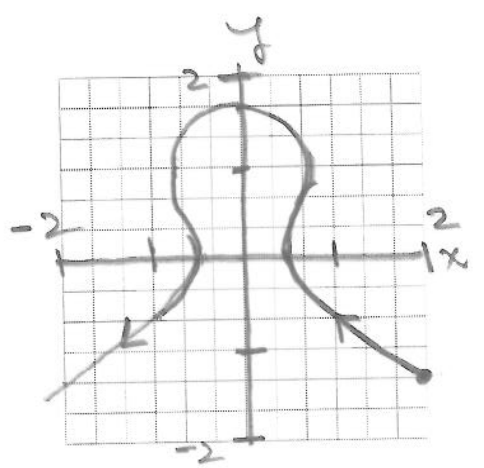
(a)





(b)





3. (15 points) Find the solution (in scalar form) to the initial-value problem

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 0, \quad y(0) = 3, \quad y'(0) = -1.$$

$$\lambda^2 + 2\lambda + 5 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$$

$$y(t) = k_1 e^{-t} \cos 2t + k_2 e^{-t} \sin 2t$$

$$y(0) = 3 \Rightarrow k_1 = 3$$

$$y(t) = 3e^{-t} \cos 2t + k_2 e^{-t} \sin 2t$$

$$y'(t) = -3e^{-t} \cos 2t - 6e^{-t} \sin 2t$$

$$- k_2 e^{-t} \sin 2t + 2k_2 e^{-t} \cos 2t$$

$$y'(0) = -3 + 2k_2 \stackrel{?}{=} -1$$

$$\Rightarrow k_2 = 1$$

$$y(t) = 3e^{-t} \cos 2t + e^{-t} \sin 2t$$

4. (16 points) Consider the system

$$\frac{dx}{dt} = 100x - x^2 - 2xy$$
$$\frac{dy}{dt} = 150y - 6y^2 - xy$$

defined on the entire xy-plane.

(a) Calculate its equilibrium points.

$$\frac{dx}{dt} = x(100 - x - 2y) \stackrel{?}{=} 0$$
$$\frac{dy}{dt} = y(150 - 6y - x) \stackrel{?}{=} 0$$

$\frac{dx}{dt} = 0$ if $x = 0$ or $x + 2y = 100$

$\frac{dy}{dt} = 0$ if $y = 0$ or $x + 6y = 150$

Equilibrium points:

$(0,0)$, $(0,25)$, $(100,0)$

and $\begin{cases} x + 2y = 100 \\ x + 6y = 150 \end{cases} \Rightarrow 4y = 50$

$y = \frac{50}{4} = \frac{25}{2}$

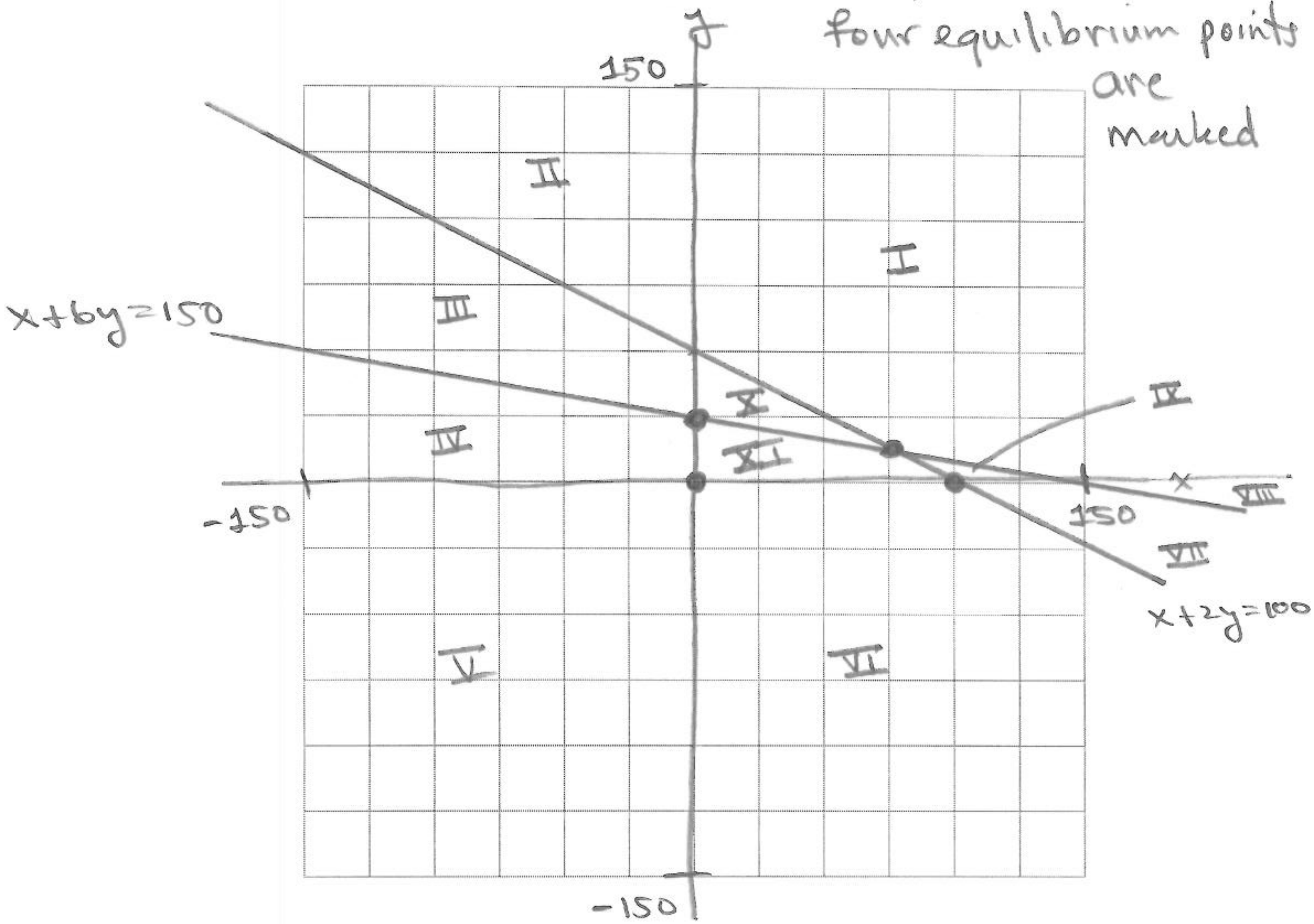
$x = 75$

$(x,y) = (75, \frac{25}{2})$

4. (continued) → four lines including x-axis and y-axis

(b) Sketch the sets on which the vector field is either horizontal or vertical. (Note that we are considering the entire xy-plane in this problem.)

four equilibrium points are marked



(c) For each region in the xy-plane bounded by the sets in part (b), indicate the direction—northeast, northwest, southeast, or southwest—in which the vector field points. (You may want to use the abbreviations NE, NW, SE, and SW respectively.)

See next page.

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Region	x	$100-x-2y$	$\frac{dx}{dt}$	y	$150-6y-x$	$\frac{dy}{dt}$	dir
I	+	−	−	+	−	−	SW
II	−	−	+	+	−	−	SE
III	−	+	−	+	−	−	SW
IV	−	+	−	+	+	+	NW
V	−	+	−	−	+	−	SW
VI	+	+	+	−	+	−	SE
VII	+	−	−	−	+	−	SW
VIII	+	−	−	−	−	+	NW
IX	+	−	−	+	+	+	NW
X	+	+	+	+	−	−	SE
XI	+	+	+	+	+	+	NE

5. (21 points) Are the following statements true or false? **You must justify your answers to receive any credit.**

(a) Two different first-order autonomous systems can have the same vector field.

False. The vector field for an autonomous system $\frac{dx}{dt} = f(x,y)$ and $\frac{dy}{dt} = g(x,y)$ is given by $F(x,y) = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$. If either $f(x,y)$ or $g(x,y)$ change, we get a different vector field.

(b) The function $\mathbf{Y}(t) = (\cos 2t, \sin t)$ is not a solution to any linear system.

True. Sinusoidal functions come from complex eigenvalues $\alpha \pm i\beta$. If $\cos 2t$ is part of a solution, then $\beta = 2$. If $\sin t$ is part of a solution, then $\beta = 1$. We cannot have both if we have a 2×2 system.

(c) If the function $(x_1(t), y_1(t)) = (\cos t, \sin t)$ is a solution of a first-order autonomous system, then the function $(x_2(t), y_2(t)) = (-\sin t, \cos t)$ is also a solution of the same system.

True. Note that $(x_2(t), y_2(t)) =$
$(\cos(t+\frac{\pi}{2}), \sin(t+\frac{\pi}{2})) = (x_1(t+\frac{\pi}{2}), y_1(t+\frac{\pi}{2}))$.
For an autonomous system, if we translate t by a given amount in one solution, we get another solution.

6. (16 points) Let

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}.$$

Compute the matrix exponential $e^{t\mathbf{A}}$. Hint: Note that $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$ where $\mathbf{D}$ is a diagonal matrix,

$$\mathbf{P} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{P}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

The matrix $\mathbf{P}^{-1}$ is the matrix such that $\mathbf{P}\,\mathbf{P}^{-1} = \mathbf{P}^{-1}\mathbf{P} = \mathbf{I}$ where $\mathbf{I}$ is the 2×2 identity matrix.

To determine D, note $D = P^{-1}AP \Rightarrow$

$$D = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

To calculate e^{tA}, we need the powers A^k of A.

$$A^k = (PD\cancel{P^{-1}})(\cancel{P}DP^{-1}) \ldots (\cancel{P}DP^{-1}) \quad (k \text{ times})$$

$$= PD^kP^{-1} = P\begin{pmatrix} 2^k & 0 \\ 0 & 3^k \end{pmatrix}P^{-1}.$$

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k = \sum_{k=0}^{\infty} \frac{t^k}{k!} PD^kP^{-1}$$

$$= P\left(\sum_{k=0}^{\infty} \frac{t^k}{k!} D^k\right)P^{-1}$$

$$= \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2e^{2t} & e^{3t} \\ e^{2t} & e^{3t} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

#6 cnt.

$$e^{tA} = \begin{pmatrix} 2e^{2t} - e^{3t} & 2e^{3t} - 2e^{2t} \\ e^{2t} - e^{3t} & 2e^{3t} - e^{2t} \end{pmatrix}$$