1. (24 points) Note that parts c and d of this problem are on the next page. Consider the second-order equation

\[ \frac{d^2 y}{dt^2} + 4 \frac{dy}{dt} + 20y = -5 \cos 5t. \]

(a) What can you say about the long-term behavior of solutions without solving for the general solution? Be as specific as possible.

The equation is damped and sinusoidally forced \( \Rightarrow \) all solutions tend to the steady-state solution. That solution has angular frequency \( \omega = 5 \) \( \Rightarrow \) period \( T = \frac{2\pi}{\omega} = \frac{2\pi}{5} \).

(b) Determine a particular solution to this differential equation.

Simplifying \( \frac{d^2 y}{dt^2} + 4 \frac{dy}{dt} + 20y = -5 e^{(5i)t} \)

Guess \( y_c(t) = a e^{(5i)t} \)

We get
\[
\begin{align*}
a((5i)^2 + 4(5i) + 20)e^{(5i)t} &= -5 e^{(5i)t} \\
a(-25+20i+20)e^{(5i)t} &= -5 e^{(5i)t}
\end{align*}
\]

\( \Rightarrow a = \frac{-5}{-5+20i} = \frac{1}{1-4i} \frac{4+4i}{17} = \frac{1+4i}{17} \)

For a solution to the original equation, we take
\[
y_p(t) = \text{Re}(a e^{(5i)t})
\]
\[
= \text{Re}(\left(\frac{1+4i}{17}\right)(\cos 5t + i \sin 5t))
\]
\[
= \left(\frac{1}{17}\right) \cos 5t - \left(\frac{4}{17}\right) \sin 5t
\]
1. (continued) Here is the differential equation from the previous page:

\[
\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 20y = -5\cos 5t.
\]

(c) Find the general solution to this differential equation.

The characteristic polynomial is \( \lambda^2 + 4\lambda + 20 \).

We get \( \lambda = -4 \pm \sqrt{16 - 80} = -2 \pm 4i \)

The general solution to the equation is

\[
y(t) = c_1 e^{-2t} \cos 4t + c_2 e^{-2t} \sin 4t \\
+ \left( \frac{1}{17} \right) \cos 5t - \left( \frac{4}{17} \right) \sin 5t
\]

(d) What can you say about the long-term behavior of the solutions given your results from parts b and c? Be as specific as possible.

The amplitude of the steady-state is

\[
|a| = \frac{1}{17} \sqrt{1 + 4^2} = \frac{\sqrt{17}}{17} = \frac{1}{\sqrt{17}}.
\]

The phase angle of the steady-state is given by \( \tan \Theta = 4 \). It is approximately 104°.

The exponential rate of approach to the steady-state is determined by the real part of the eigenvalue, which is -2.
2. (16 points) For what values of $a$, $b$, $c$, and $d$ is the linear system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbf{Y}$$

Hamiltonian? Calculate a Hamiltonian function for these values.

We want $H(x,y)$ such that

$$\frac{dx}{dt} = ax + by = \frac{\partial H}{\partial y}$$
$$\frac{dy}{dt} = cx + dy = -\frac{\partial H}{\partial x}$$

Therefore, we want

$$\frac{\partial}{\partial x}(ax+by) = -\frac{\partial}{\partial y}(cx+dy)$$

$$\Rightarrow a = -d$$

In this case, our system becomes

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx - ay$$

To compute $H(x,y) = \int (ax+by)\,dy + \phi(x)$

$$= axy + b\frac{x^2}{2} + \phi(x)$$

Then $cx - ay = -(ay + \phi'(x)) \Rightarrow \phi'(x) = -cx$

$$\Rightarrow \phi(x) = -\frac{cx^2}{2}$$

We have $H(x,y) = \frac{by^2}{2} + axy - \frac{cx^2}{2}$.
3. (20 points) Note that part c of this problem is on the next page.

(a) Calculate \( \mathcal{L}^{-1} \left[ \frac{2s + 5}{s^2 + 2s + 3} \right] \).

\[
\frac{2s + 5}{(s + 1)^2 + 2} = \frac{2(s + 1)}{(s + 1)^2 + 2} + \frac{3}{(s + 1)^2 + 2}
\]

\[
\mathcal{L}^{-1} \left[ \frac{2s + 5}{s^2 + 2s + 3} \right] = 2e^{-t} \cos \sqrt{2}t + \frac{3}{\sqrt{2}}e^{-t} \sin \sqrt{2}t
\]

(b) Calculate the Laplace transform \( \mathcal{L}[y] \) for the solution \( y(t) \) to the initial-value problem

\[
\frac{d^2 y}{dt^2} + 2 \frac{dy}{dt} + 3y = \delta(t), \quad y(0) = 2, \quad y'(0) = 1.
\]

DO NOT CALCULATE A FORMULA FOR \( y(t) \) HERE.

\[
(s^2 \mathcal{L}[y] - 2s - 1) + 2(s \mathcal{L}[y] - 2) + 3 \mathcal{L}[y] = e^{-4s}
\]

\[
(s^2 + 2s + 3) \mathcal{L}[y] = 2s + 5 + e^{-4s}
\]

\[
\mathcal{L}[y] = \frac{2s + 5}{s^2 + 2s + 3} + \frac{e^{-4s}}{s^2 + 2s + 3}
\]
3. (continued)

(c) Calculate the solution $y(t)$ to the initial-value problem in part b.

\[
L^{-1}\left[ \frac{1}{s^2+2s+3} \right] = L^{-1}\left[ \frac{1}{(s+1)^2+2} \right] = \frac{1}{\sqrt{2}} \, e^{-t} \sin \sqrt{2}t
\]

\[
\Rightarrow
L^{-1}\left[ \frac{e^{-4s}}{s^2+2s+3} \right] = \frac{1}{\sqrt{2}} \, u_4(t) \, e^{-(t-4)} \sin \sqrt{2}(t-4)
\]

We obtain the solution

\[
y(t) = 2e^{-t} \cos \sqrt{2}t + \frac{3}{\sqrt{2}} \, e^{-t} \sin \sqrt{2}t
\]

\[
+ \frac{1}{\sqrt{2}} \, u_4(t) \, e^{-(t-4)} \sin \sqrt{2}(t-4)
\]
4. (16 points) Consider the one-parameter family of linear systems

\[
\frac{dY}{dt} = \begin{pmatrix} a & 1 \\ a & a \end{pmatrix} Y.
\]

(a) Sketch the curve in the trace-determinant plane that is obtained by varying the parameter \(a\).

\[
T = \text{trace} = 2a \\
D = \text{det} = a^2 - a \\
D = \left(\frac{T}{2}\right)^2 - \frac{T}{2} \\
= \frac{T^2}{4} - \frac{T}{2} \\
= \frac{1}{4}(T^2 - 2T) \quad \text{parabola}
\]

(b) Determine all bifurcation values of \(a\) and briefly discuss the different types of phase portraits that are exhibited in this one-parameter family.

Need to determine where the two parabolas intersect.

\[
D = \frac{T^2}{4} = \frac{1}{4}(T^2 - 2T) \\
\Rightarrow T^2 = T^2 - 2T \\
\Rightarrow 0 = -2T \\
\Rightarrow T = 0
\]

For \(T = 2a\) negative, the system is a spiral sink.
For \(a = 0\), we have a bifurcation value.
For \(0 < T < 2\) (equivalently \(0 < a < 1\)),
the system is a saddle.
For \(T = 2\) (\(\Rightarrow a = 1\)), we have another bifurcation value. For \(T > 2\) (\(a > 1\)),
the system is a source with real eigenvalues.
5. (24 points) Note that part c of this problem is on the next page. Consider the system
\[
\frac{dx}{dt} = x - 3y^2 \\
\frac{dy}{dt} = x - 3y - 6.
\]
(a) Sketch the nullclines and indicate the directions in which solutions cross the nullclines.

\[
\frac{dx}{dt} = 0 \iff x = 3y^2 \\
\frac{dy}{dt} = 0 \iff 3y = x - 6 \iff y = \frac{1}{3}x - 2
\]

Nullclines intersect if
\[
3y^2 - 3y + 6 = 0 \Rightarrow y^2 - y - 2 = 0 \Rightarrow (y - 2)(y + 1) = 0 \Rightarrow y = 2 \text{ or } -1
\]

(b) Find and classify all equilibrium points.

\[
J(x, y) = \begin{pmatrix} 1 & -6y \\ 1 & -3 \end{pmatrix}
\]

\[
J(12, 2) = \begin{pmatrix} 1 & -12 \\ 1 & -3 \end{pmatrix} \Rightarrow \lambda^2 + 2\lambda + 9 = 0 \Rightarrow \lambda = -2 \pm \sqrt{4 - 36} = -2 \pm \sqrt{32}
\]

\Rightarrow (12, 2) is a spiral sink

\[
J(3, -1) = \begin{pmatrix} 1 & 6 \\ 1 & -3 \end{pmatrix}
\]

From the negative determinant, we know that (3, -1) is a saddle.
5. (continued)

(c) Sketch the phase portrait.