1. (24 points) **Note that parts c and d of this problem are on the next page.** Consider the second-order equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 20y = -5\cos 5t.$$

(a) What can you say about the long-term behavior of solutions without solving for the general solution? Be as specific as possible.

The equation is damped and sinusoidally forced $\Rightarrow$ all solutions tend to the steady-state solution That solution has angular frequency $= 5 \Rightarrow$ period $= \frac{2\pi}{5}$.

(b) Determine a particular solution to this differential equation.

Complexify: $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 20y = -5e^{(5i)t}$

Guess $y_c(t) = ae^{(5i)t}$

We get

$a((5i)^2 + 4(5i) + 20)e^{(5i)t} \stackrel{?}{=} -5e^{(5i)t}$

$a(-25 + 20i + 20)e^{(5i)t} \stackrel{?}{=} -5e^{(5i)t}$

$\Rightarrow a = \frac{-5}{-5+20i} = \frac{1}{1-4i}\frac{(1+4i)}{(1+4i)} = \frac{1+4i}{17}$

For a solution to the original equation, we take

$y_p(t) = \mathrm{Re}(ae^{(5i)t})$

$= \mathrm{Re}\left(\left(\frac{1+4i}{17}\right)(\cos 5t + i\sin 5t)\right)$

$= \left(\frac{1}{17}\right)\cos 5t - \left(\frac{4}{17}\right)\sin 5t$

1. (continued) Here is the differential equation from the previous page:

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 20y = -5\cos 5t.$$

(c) Find the general solution to this differential equation.

The characteristic polynomial is $\lambda^2 + 4\lambda + 20$.

We get $\lambda = \dfrac{-4 \pm \sqrt{16-80}}{2} = -2 \pm 4i$

The general solution to the equation is

$y(t) = c_1 e^{-2t} \cos 4t + c_2 e^{-2t} \sin 4t$

$+ \left(\frac{1}{17}\right) \cos 5t - \left(\frac{4}{17}\right) \sin 5t$

(d) What can you say about the long-term behavior of the solutions given your results from parts b and c? Be as specific as possible.

The amplitude of the steady-state is

$|a| = \frac{1}{17}|1+4i| = \frac{\sqrt{17}}{17} = \frac{1}{\sqrt{17}}$.

The phase angle of the steady-state is given by $\tan\theta = 4$. It is approximately 104^0.

The exponential rate of approach to the steady-state is determined by the real part of the eigenvalue, which is -2.

2. (16 points) For what values of a, b, c, and d is the linear system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbf{Y}$$

Hamiltonian? Calculate a Hamiltonian function for these values.

We want $H(x,y)$ such that

$$\frac{dx}{dt} = ax + by = \frac{\partial H}{\partial y}$$

$$\frac{dy}{dt} = cx + dy = -\frac{\partial H}{\partial x}.$$

Therefore, we want

$$\frac{\partial(ax+by)}{\partial x} = -\frac{\partial(cx+dy)}{\partial y}$$

$$\Rightarrow a = -d.$$

In this case, our system becomes

$$\frac{dx}{dt} = ax + by$$

$$\frac{dy}{dt} = cx - ay.$$

To compute $H(x,y) = \int (ax+by)\,dy + \varphi(x)$

$$= axy + b\frac{y^2}{2} + \varphi(x).$$

Then $cx - ay = -(ay + \varphi'(x)) \Rightarrow \varphi'(x) = -cx$

$$\Rightarrow \varphi(x) = -c\frac{x^2}{2}$$

We have $H(x,y) = \frac{by^2}{2} + axy - \frac{cx^2}{2}.$

3. (20 points) **Note that part c of this problem is on the next page.**

(a) Calculate $\mathcal{L}^{-1}\left[\dfrac{2s+5}{s^2+2s+3}\right]$.

$$\frac{2s+5}{(s+1)^2+2} = \frac{2(s+1)}{(s+1)^2+2} + \frac{3}{(s+1)^2+2}$$

$$\mathcal{L}^{-1}\left[\frac{2s+5}{s^2+2s+3}\right] = 2e^{-t}\cos\sqrt{2}t + \frac{3}{\sqrt{2}}e^{-t}\sin\sqrt{2}t$$

(b) Calculate the Laplace transform $\mathcal{L}[y]$ for the solution $y(t)$ to the initial-value problem

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 3y = \delta_4(t), \quad y(0) = 2, \quad y'(0) = 1.$$

DO NOT CALCULATE A FORMULA FOR y(t) HERE.

$$\left(s^2\mathcal{L}[y] - 2s - 1\right) + 2\left(s\mathcal{L}[y] - 2\right) + 3\mathcal{L}[y] = e^{-4s}$$

$$(s^2+2s+3)\mathcal{L}[y] = 2s+5 + e^{-4s}$$

$$\mathcal{L}[y] = \frac{2s+5}{s^2+2s+3} + \frac{e^{-4s}}{s^2+2s+3}$$

3. (continued)

(c) Calculate the solution $y(t)$ to the initial-value problem in part b.

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+2s+3}\right] = \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2+2}\right] = \frac{1}{\sqrt{2}} e^{-t} \sin\sqrt{2}t$$

$$\Longrightarrow$$

$$\mathcal{L}^{-1}\left[\frac{e^{-4s}}{s^2+2s+3}\right] = \frac{1}{\sqrt{2}} u_4(t)\, e^{-(t-4)} \sin\sqrt{2}(t-4)$$

We obtain the solution

$$y(t) = 2e^{-t}\cos\sqrt{2}t + \frac{3}{\sqrt{2}} e^{-t} \sin\sqrt{2}t + \frac{1}{\sqrt{2}} u_4(t)\, e^{-(t-4)} \sin\sqrt{2}(t-4)$$

4. (16 points) Consider the one-parameter family of linear systems

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} a & 1 \\ a & a \end{pmatrix} \mathbf{Y}.$$

(a) Sketch the curve in the trace-determinant plane that is obtained by varying the parameter a.

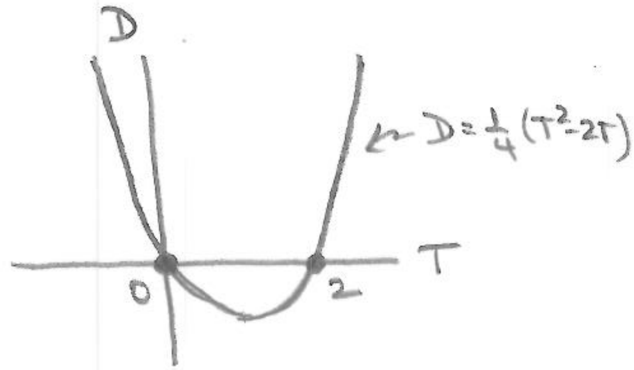
$T = \text{trace} = 2a$

$D = \det = a^2 - a$

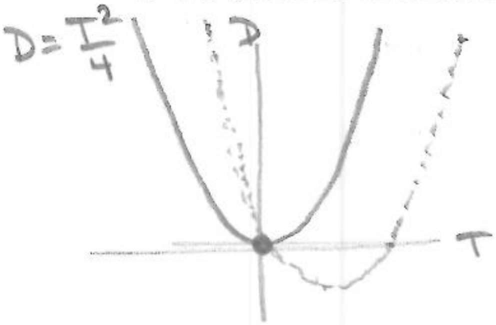
$D = \left(\frac{T}{2}\right)^2 - \frac{T}{2}$

$= \frac{T^2}{4} - \frac{T}{2}$

$= \frac{1}{4}(T^2 - 2T)$ parabola



(b) Determine all bifurcation values of a and briefly discuss the different types of phase portraits that are exhibited in this one-parameter family.



Need to determine where the two parabolas intersect.

$D = \frac{T^2}{4} = \frac{1}{4}(T^2 - 2T)$

$\Rightarrow T^2 = T^2 - 2T \Rightarrow 0 = -2T$

$\Rightarrow T = 0$

For $T = 2a$ negative, the system is a spiral sink.

For $a = 0$, we have a bifurcation value.

For $0 < T < 2$ (equivalently $0 < a < 1$), the system is a saddle.

For $T = 2$ ($\Rightarrow a = 1$), we have another bifurcation value. For $T > 2$ ($a > 1$), the system is a source with real eigenvalues.

5. (24 points) **Note that part c of this problem is on the next page.** Consider the system

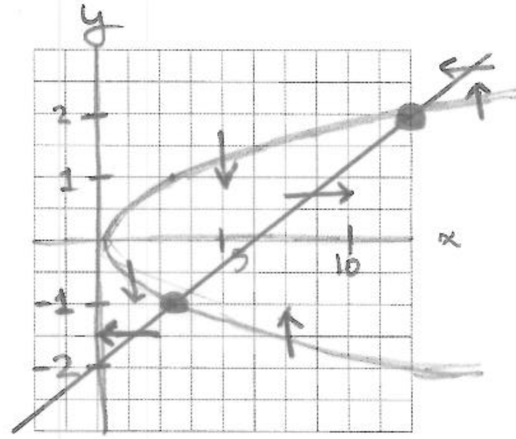
$$\frac{dx}{dt} = x - 3y^2$$
$$\frac{dy}{dt} = x - 3y - 6.$$

(a) Sketch the nullclines and indicate the directions in which solutions cross the nullclines.

$\frac{dx}{dt} = 0 \iff x = 3y^2$

$\frac{dy}{dt} = 0 \iff 3y = x - 6$

$\iff y = \frac{1}{3}x - 2$



Nullclines intersect if

$3y^2 = 3y + 6 \Rightarrow$

$3y^2 - 3y - 6 = 0 \Rightarrow y^2 - y - 2 = 0 \Rightarrow (y-2)(y+1) = 0$

$\Rightarrow y = 2$ or -1

(b) Find and classify all equilibrium points.

$(x,y) = (12,2)$ or $(3,-1)$

$J(x,y) = \begin{pmatrix} 1 & -6y \\ 1 & -3 \end{pmatrix}$

$J(12,2) = \begin{pmatrix} 1 & -12 \\ 1 & -3 \end{pmatrix} \Rightarrow \lambda^2 + 2\lambda + 9 = 0$

$\lambda = \frac{-2 \pm \sqrt{4 - 36}}{2}$

$\Rightarrow (12,2)$ is a spiral sink

$J(3,-1) = \begin{pmatrix} 1 & 6 \\ 1 & -3 \end{pmatrix}$

From the negative determinant, we know that $(3,-1)$ is a saddle.

5. (continued)

(c) Sketch the phase portrait.