1. (14 points) Consider the second-order equation

$$\frac{d^2y}{dt^2} + 4y = 3\cos 2t. \qquad (*)$$

(a) Determine a particular solution to this differential equation.

resonance - complexify $\frac{d^2y}{dt^2} + 4y = 3e^{2it}$

Guess $y_c = a t e^{2it} \Rightarrow \frac{dy_c}{dt} = a(1+2it)e^{2it}$

$\Rightarrow \frac{d^2y_c}{dt^2} = a(2i - 4t + 2i)e^{2it} = 4a(-t+i)e^{2it}$

$\frac{d^2y_c}{dt^2} + 4y = 4a(\cancel{-t}+i)e^{2it} + \cancel{4ate^{2it}}$

$= (4a)(i)e^{2it} \stackrel{?}{=} 3e^{2it}$

$\Rightarrow a = \frac{3}{4i} = -\frac{3}{4}i$

$y_p = \mathrm{Re}\, y_c = \mathrm{Re}\left(-\frac{3}{4}i t e^{2it}\right)$

$= +\frac{3}{4}t\sin 2t$

(b) Determine the general solution of this differential equation.

general solution of assoc. homogeneous eq. $k_1\cos 2t + k_2 \sin 2t$

general solution of $(*)$ is

$k_1\cos 2t + k_2\sin 2t + \frac{3}{4}t\sin 2t$

2. (20 points) **Note that Part (c) of this problem is on the next page.** Consider the linear system

$$\frac{dx}{dt} = -5x - 2y$$
$$\frac{dy}{dt} = -x - 4y.$$

$$\frac{dY}{dt} = \begin{pmatrix} -5 & -2 \\ -1 & -4 \end{pmatrix} Y$$

(a) Determine the type of the equilibrium point at the origin and find all straight-line solutions. Make sure that you show the computations that justify your answers.

char. poly $\lambda^2 + 9\lambda + 18 = (\lambda+3)(\lambda+6)$

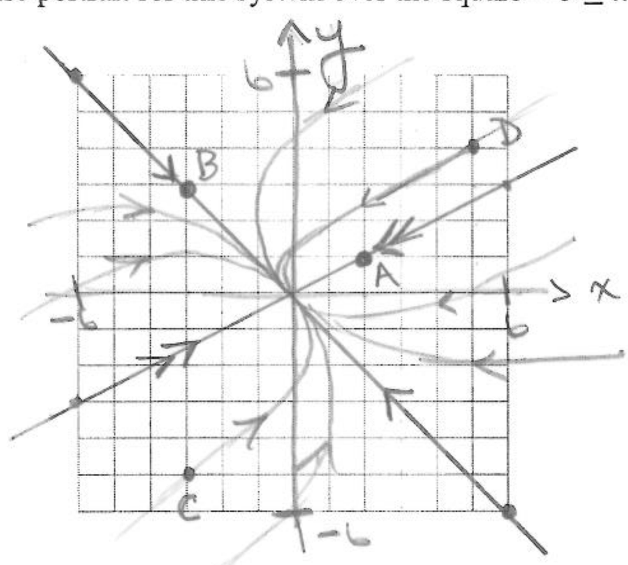
$\lambda = -3$ and $-6 \Rightarrow$ sink at origin

$\lambda = -3$ evecs: $\begin{cases} -5x - 2y = -3x \\ -x - 4y = -3y \end{cases}$ $y = -x$ slow

$\lambda = -6$ evecs: $\begin{cases} -5x - 2y = -6x \\ -x - 4y = -6y \end{cases}$ $2y = x$ fast

$Y(t) = k_1 e^{-3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $Y(t) = k_2 e^{-6t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

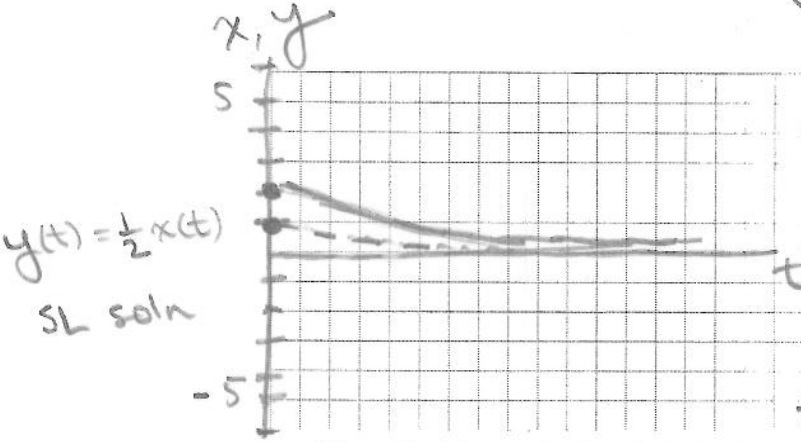
(b) Sketch the phase portrait for this system over the square $-6 \le x \le 6$ and $-6 \le y \le 6$.



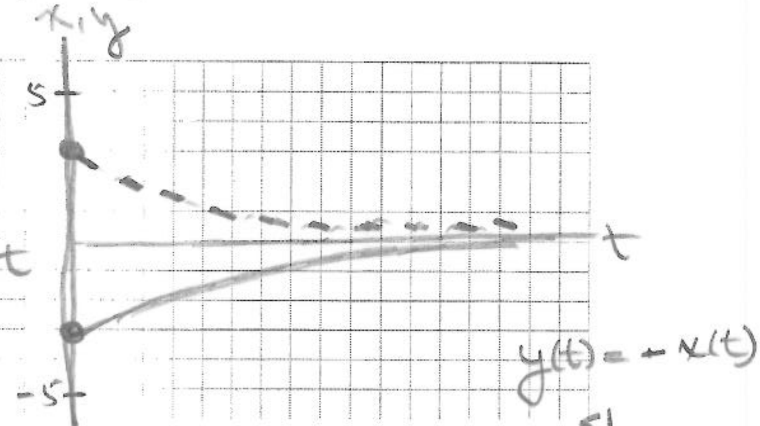
2. (continued)

(c) Plot the initial conditions $A = (2, 1)$, $B = (-3, 3)$, $C = (-3, -5)$, and $D = (5, 4)$ on your phase portrait on the previous page. Then plot the $x(t)$- and $y(t)$-graphs for $t \geq 0$ for the initial conditions A, B, C, and D below. Make sure that you label the axes, indicate a scale on the vertical axis, and distinguish the $x(t)$- from the $y(t)$-graph.

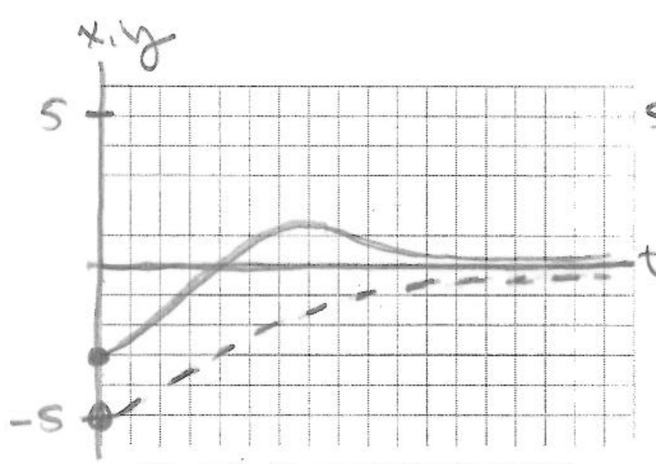
— = x(t)-graph
- - - = y(t)-graph



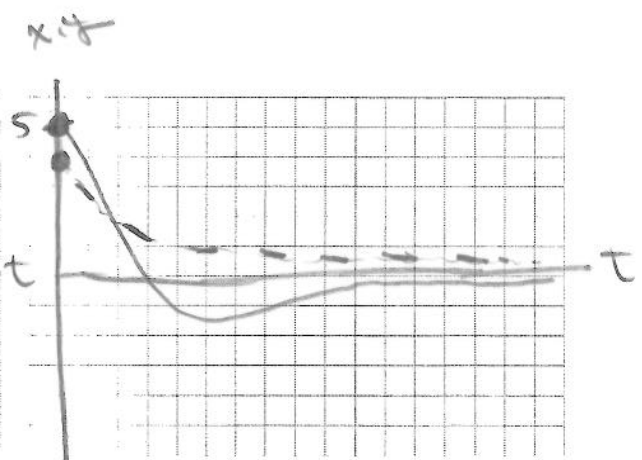
$x(t)$- and $y(t)$-graphs for the initial condition A



$x(t)$- and $y(t)$-graphs for the initial condition B



$x(t)$- and $y(t)$-graphs for the initial condition C



$x(t)$- and $y(t)$-graphs for the initial condition D

3. (12 points) Consider the differential equation

$$\frac{dy}{dt} = y^2 - (2t+3)y + t^2 + 3t + 1.$$

(a) Show that the functions $y_1(t) = t$ and $y_2(t) = t + 3$ are solutions of this differential equation. (Hint: Do not waste your time trying to come up with the general solution.)

To check $y_1(t)$:

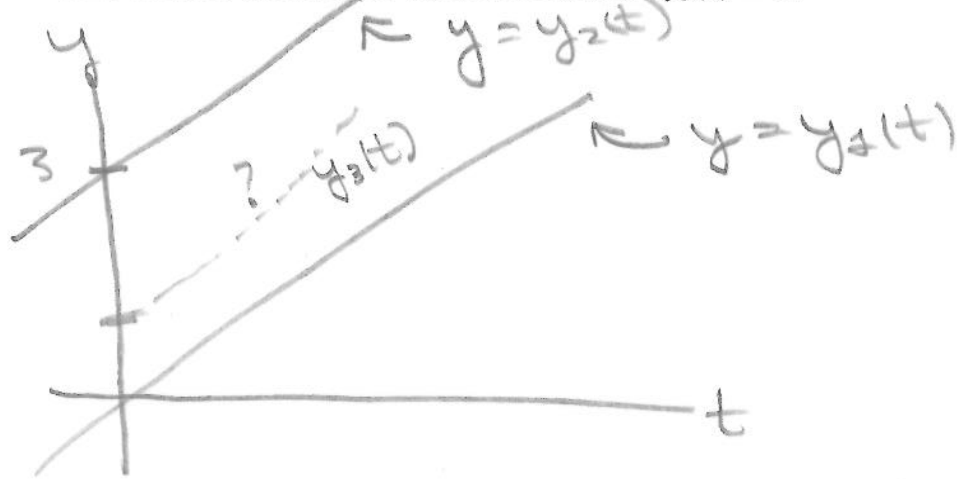
$$\frac{dy_1}{dt} = 1 \stackrel{?}{=} y_1^2 - (2t+3)y_1 + t^2 + 3t + 1$$
$$\stackrel{?}{=} t^2 - (2t+3)t + t^2 + 3t + 1$$
$$\stackrel{?}{=} \cancel{t^2} - \cancel{2t^2} - \cancel{3t} + \cancel{t^2} + \cancel{3t} + 1$$

To check $y_2(t)$:

$$\frac{dy_2}{dt} = 1 \stackrel{?}{=} y_2^2 - (2t+3)y_2 + t^2 + 3t + 1$$
$$\stackrel{?}{=} (t+3)^2 - (2t+3)(t+3) + t^2 + 3t + 1$$

see next page

(b) Using the Existence and Uniqueness Theorem, what can you say about $y_3(1)$ if $y_3(t)$ is the solution that satisfies the initial condition $y_3(0) = 1$?



Uniqueness $\Rightarrow$ $y_3(1)$ satisfies

$y_1(1) < y_3(1) < y_2(1)$

$1 < y_3(1) < 4$

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#3) (a) cnt.

$(t+3)^2 - (2t+3)(t+3) + t^2 + 3t + 1 =$

$\cancel{t^2} + \cancel{6t} + \cancel{9} - \cancel{2t^2} - \cancel{9t} - \cancel{9} + \cancel{t^2} + \cancel{3t} + 1 =$

1

$\Rightarrow \frac{dy_2}{dt} = 1 \stackrel{?}{=} 1$

$\Rightarrow y_2(t) = t+3$ is a solution to

$$\frac{dy}{dt} = y^2 - (2t+3)y + t^2 + 3t + 1$$

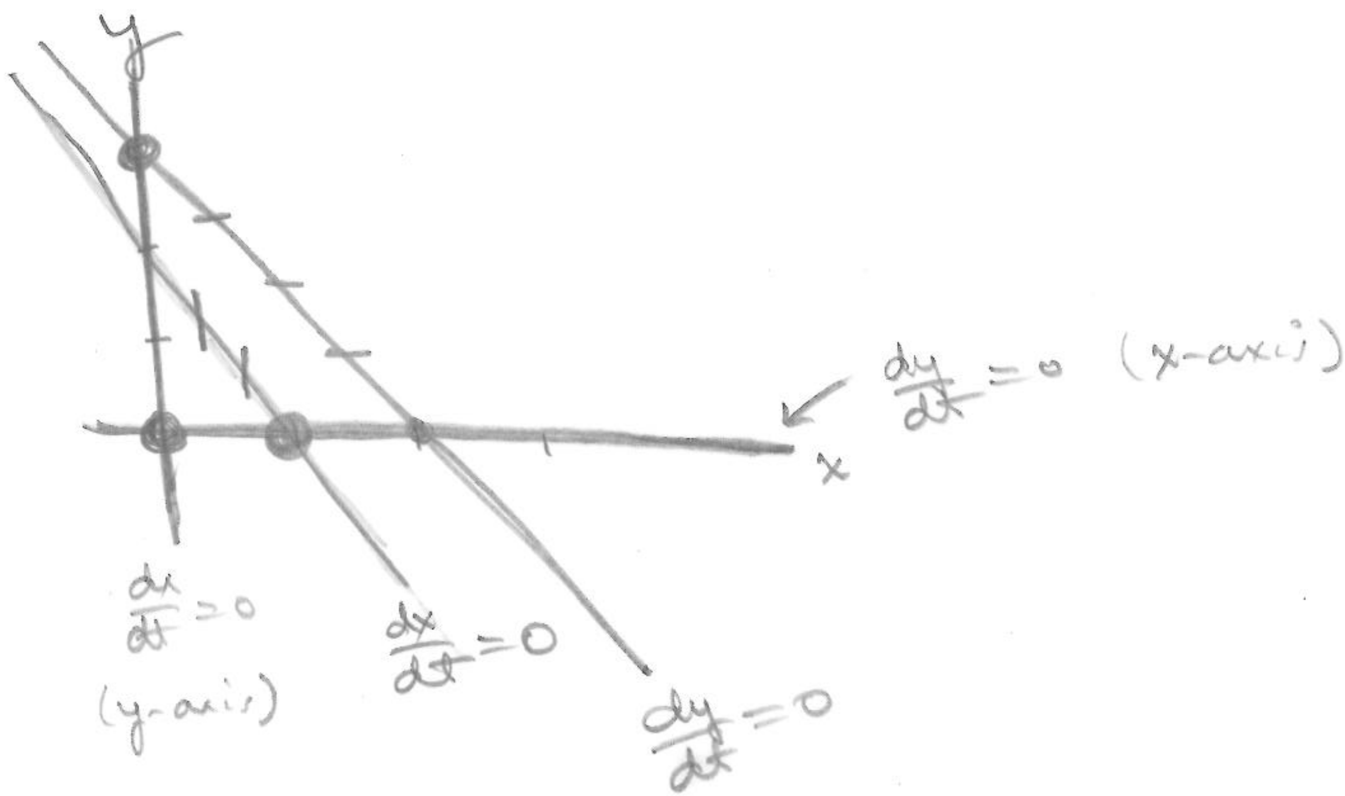
4. (20 points) **Note that Parts (b) and (c) of this problem are on the next page.** Let x represent a population of bozos and let y represent a population of barneys. These populations compete for a common food source and can be modeled by the system of differential equations

$$\frac{dx}{dt} = 2x - 2x^2 - xy$$
$$\frac{dy}{dt} = 6y - 3xy - 2y^2.$$

(a) Sketch the nullclines and determine all of the equilibrium points of this system.

$$\frac{dx}{dt} = (2-2x-y)x$$

$$\frac{dy}{dt} = (6-3x-2y)y$$



eq. points: $(0,0)$, $(1,0)$, $(0,3)$

Problem 4 (continued):

(b) Identify the types of the equilibrium points that you found in Part (a). In other words, determine if they are sinks, centers, saddles or sources. Make sure that you indicate how you derived your answer.

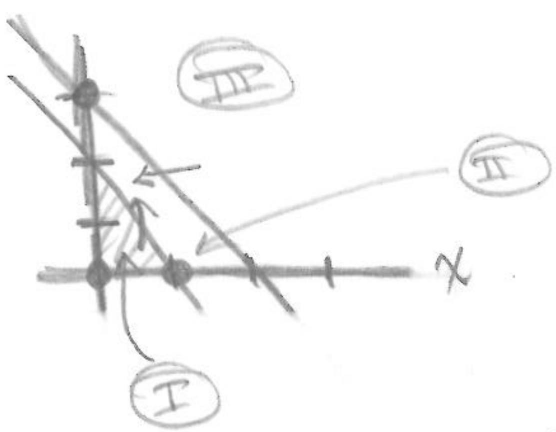
Jacobian: $J(x,y) = \begin{pmatrix} 2-4x-y & -x \\ -3y & 6-3x-4y \end{pmatrix}$

$J(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}$ source

$J(1,0) = \begin{pmatrix} -2 & 1 \\ 0 & 3 \end{pmatrix}$ saddle

$J(0,3) = \begin{pmatrix} -1 & 0 \\ -9 & -6 \end{pmatrix}$ sink

(c) What will happen to these two populations over the long term? Make sure that you provide a rigorous mathematical justification for your answer.

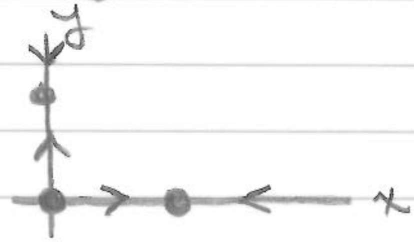


In ①, solutions head NE. In ②, solutions go NW. In ③, solutions head SW.

→ cont.

#4) c) cont.

We need only consider initial conditions in the 1st quadrant. Along the axes we have



Solutions with initial conditions in region I must enter region II. Solutions in region II head north-west and approach the equilibrium point at (0,3). They cannot enter region III.

Solutions with initial conditions in region III head south-west. They either approach the equilibrium point at (0,3) or they enter region II.

Therefore, all solutions with initial conditions in the 1st quadrant (not on axes) approach the equilibrium point (0,3). The bozos go extinct.

5. (16 points) **Note that Parts (b) and (c) of this problem are on the next page.**

   (a) Using only the definition of the Laplace transform, calculate the transform $\mathcal{L}[u_a]$, where $u_a(t)$ is the Heaviside function that "turns on" at $t = a$. Specify the domain of $\mathcal{L}[u_a]$.

$$\mathcal{L}[u_a] = \int_0^\infty u_a(t)\, e^{-st}\, dt$$

$$= \int_a^\infty e^{-st}\, dt$$

$$= \lim_{b\to\infty} \int_a^b e^{-st}\, dt$$

$$= \lim_{b\to\infty} \left[-\frac{1}{s} e^{-st}\right]_a^b$$

$$= -\frac{1}{s} \lim_{b\to\infty} \left[e^{-st}\right]_a^b$$

$$= -\frac{1}{s} \lim_{b\to\infty} \left[e^{-sb} - e^{-sa}\right]$$

$$= \frac{e^{-sa}}{s} \quad \text{if } s > 0.$$

Problem 5 (continued):

(b) Use Laplace transforms to solve the initial-value problem

$$\frac{dy}{dt} + 3y = u_4(t), \quad y(0) = 5.$$

Apply $\mathcal{L}[\;]$ to the equation:

$$s\mathcal{L}[y] - 5 + 3\mathcal{L}[y] = \frac{e^{-4s}}{s}$$

$$(s+3)\mathcal{L}[y] = 5 + \frac{e^{-4s}}{s}$$

$$\mathcal{L}[y] = \frac{5}{s+3} + \frac{e^{-4s}}{s(s+3)}$$

$$\frac{1}{s(s+3)} = \left(\frac{1}{3}\right)\left(\frac{1}{s}\right) - \left(\frac{1}{3}\right)\left(\frac{1}{s+3}\right)$$

$$\mathcal{L}^{-1}\left[\frac{1}{s(s+3)}\right] = \frac{1}{3} - \frac{1}{3}e^{-3t}$$

(cont) $\longrightarrow$

(c) Laplace transforms tend to be applied only to a certain type of differential equation. What type is that? Why?

Laplace transforms are applied to linear differential equations because we have nice formulas for $\mathcal{L}[\frac{dy}{dt}]$ and $\mathcal{L}[\;]$ is "linear", but we do not have nice formulas for $\mathcal{L}[y^2]$ in terms of $\mathcal{L}[y]$.

#5) (b) cont.

$$\mathcal{L}^{-1}\left[\frac{e^{-4s}}{s(s+3)}\right] = u_4(t)\left(\tfrac{1}{3}\right)\left(1-e^{-3(t-4)}\right)$$

$$\Rightarrow y(t) = 5e^{-3t} + \tfrac{1}{3}u_4(t)\left(1-e^{-3(t-4)}\right)$$

6. (18 points) **Note that Part (c) of this problem is on the next page.** A simple model of the learning process is based on the assumption:

   The rate at which a particular body of knowledge is learned is proportional to the fraction of material that remains to be learned.

   (a) Formulate a differential equation that models this assumption. You may want to represent the fraction of material learned as a percentage or as a quantity between 0 (= no knowledge) and 1 (= all information learned). In any case, be precise about what your variables mean.

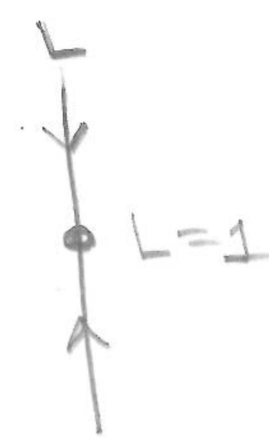
   L = % of knowledge $(0 \le L \le 1)$

   $\frac{dL}{dt} = k(1 - L)$ k constant of proportionality

   t = time

   (b) Do a qualitative analysis of the model that you derived in Part (a), i.e., sketch the phase line, etc. What will happen over the long term?

   $\frac{dL}{dt} = k(1-L)$ autonomous

   Eq. point $L = 1$

   $\frac{dL}{dt} > 0$ if $L < 1$

   $\frac{dL}{dt} < 0$ if $L > 1$

   Equilibrium solution $L = 1$ is a sink and all solutions tend to this eq. solution as $t \to \infty$.

Problem 6 (continued):

(c) Suppose that the constant of proportionality in your model in Part (a) is 0.1. How long does it take someone to learn 90% of a given topic assuming that they start with no knowledge of the topic?

$$\frac{dL}{dt} = \frac{1}{10}(1-L)$$ separable linear

$$\int\left(\frac{1}{1-L}\right)dL = \int\left(\frac{1}{10}\right)dt$$

$$-\ln(1-L) = \frac{1}{10}t + c_1$$

$$\ln(1-L) = -\frac{1}{10}t + c_2$$

$$1-L = c_3 e^{-t/10}$$

$$L = 1 - c_3 e^{-t/10}$$

$$L(0)=0 \Rightarrow L = 1 - e^{-t/10}$$

want t such that $L = .9$

$$.9 = 1 - e^{-t/10} \Rightarrow e^{-t/10} = .1$$

$$-t/10 = \ln .1 \Rightarrow t/10 = \ln 10$$

$$\Rightarrow t = 10 \ln 10.$$