More on the guessing technique for the damped harmonic oscillator

Last class we guessed that solutions of the form $y(t) = e^{\lambda t}$ might be solutions to the damped harmonic oscillator

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0.$$

In fact, we saw that $y(t) = e^{\lambda t}$ is a solution if and only if

$$m\lambda^2 + b\lambda + k = 0.$$

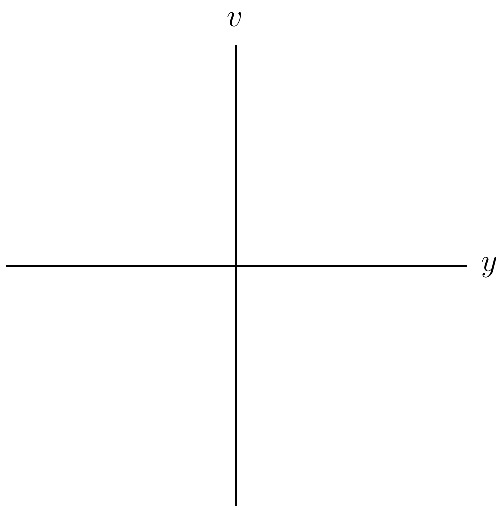
This equation is called the *characteristic equation* of the harmonic oscillator, and the polynomial $p(\lambda) = m\lambda^2 + b\lambda + k$ is its *characteristic polynomial*.

Example. Consider the harmonic oscillator

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0.$$

Its characteristic equation is

Let's plot these solutions with `HPGSystemSolver`. What are the corresponding solution curves and component graphs?



Euler's method for a system

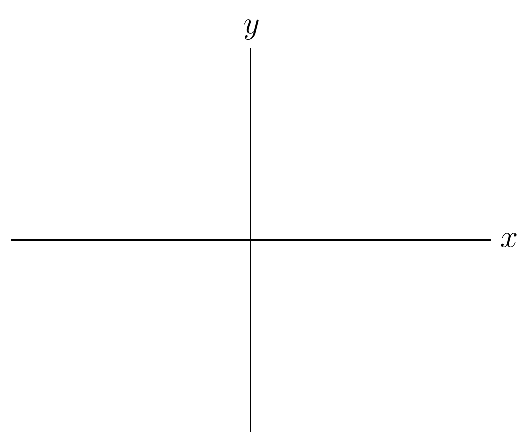
We can use the vector field for a system to produce numerical approximations for the solutions.

Example. Consider the IVP

$$\begin{aligned} \frac{dx}{dt} &= -y \\ \frac{dy}{dt} &= x - y \end{aligned} \qquad (x_0, y_0) = (2, 0).$$

The `EulersMethodForSystems` tool demonstrates the method. We pick a large step size $\Delta t = 0.5$ so that we can see the method in action.

k	x_k	y_k	m_k	n_k
0	2	0		
1				
2				
3				
4				
5				
6				



Now let's derive the general equations for Euler's method for an autonomous IVP of the form

$$\begin{aligned}\frac{dx}{dt} &= f(x,y)\\ \frac{dy}{dt} &= g(x,y)\end{aligned} \qquad (x(t_0), y(t_0)) = (x_0, y_0).$$

Euler's method for systems is just as easy to program as Euler's method for equations. Once again here's how we can program it with a spreadsheet.

	A	B	C	D	E	F	G
0	0	2	0	0.5			
1							
2							
3							
4							
5							
6							
7							
8							
9							
10							
11							
12							
13							
14							
15							
16							
17							
18							

There are two spreadsheets posted on the course web site—one for the example above and one for the following example.

Example. Consider the predator-prey system

$$\frac{dR}{dt} = R - 0.2RF$$
$$\frac{dF}{dt} = -0.3F + 0.1RF$$

along with the initial condition $(R_0, F_0) = (1, 2)$. Using the spreadsheet on the web site, we see that Euler's method has trouble approximating periodic solutions.

`HPGSystemSolver` uses a more sophisticated fixed-step-size algorithm called the Runge-Kutta method. It usually works better than Euler's method, but there are equations for which any fixed-step-size algorithm is not appropriate.

Existence and Uniqueness Theory for Systems

There is an existence and uniqueness theorem for systems just like the theorem for equations.

Existence and Uniqueness Theorem. Let

$$\frac{d\mathbf{Y}}{dt} = \mathbf{F}(t, \mathbf{Y})$$

be a system of differential equations. Suppose that t_0 is an initial time and $\mathbf{Y}_0$ is an initial value. Suppose also that the function $\mathbf{F}$ is continuously differentiable. Then there is an $\epsilon > 0$ and a function $\mathbf{Y}(t)$ defined for $t_0 - \epsilon < t < t_0 + \epsilon$ such that

$$\frac{d\mathbf{Y}}{dt} = \mathbf{F}(t, \mathbf{Y}(t)) \quad \text{and} \quad \mathbf{Y}(t_0) = \mathbf{Y}_0.$$

In other words, $\mathbf{Y}(t)$ satisfies the initial-value problem. Moreover, for t in this interval, this solution is unique.

There is an important consequence of the Uniqueness Theorem for autonomous systems: Consider the metaphor of the parking lot.

Given the autonomous system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{F}(\mathbf{Y}).$$

Let $\mathbf{Y}_0$ be an initial condition such that $\mathbf{Y}_1(t)$ is a solution that satisfies $\mathbf{Y}(t_1) = \mathbf{Y}_0$ and $\mathbf{Y}_2(t)$ is another solution that satisfies $\mathbf{Y}(t_2) = \mathbf{Y}_0$. Then

$$\mathbf{Y}_2(t) = \mathbf{Y}_1(t - (t_2 - t_1)).$$

Example. Consider the second-order equation

$$\frac{d^2y}{dt^2} + y = 0,$$

which is equivalent to the system

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -y. \end{aligned}$$

Note that

$$\mathbf{Y}_1(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

are both solutions to the system. How are $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ related?

There is an animation on the web site that illustrates this phenomenon.

Here is an informal restatement of this consequence of uniqueness:

> For an autonomous system, if two solution curves in the phase plane touch, then they are identical.