Linear systems

Linear systems and second-order linear equations are the most important systems we study in this course.

What is a linear system with two dependent variables?

What is a second-order, homogeneous, linear equation?

How do we write linear systems in vector notation?

Recall two examples that we have already discussed.

Example 1. We have already calculated the general solution to the partially decoupled system

$$\begin{aligned} \frac{dx}{dt} &= 2y - x \\ \frac{dy}{dt} &= y. \end{aligned}$$

It is

$$\begin{aligned} x(t) &= y_0 e^t + (x_0 - y_0)e^{-t} \\ y(t) &= y_0 e^t. \end{aligned}$$

Example 2. For the damped harmonic oscillator

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0,$$

we found two (scalar) solutions

$$y_1(t) = e^{-t} \qquad \text{and} \qquad y_2(t) = e^{-2t}.$$

You should also recall that this second-order equation can be converted to a first-order system where

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -2y - 3v. \end{aligned}$$

Given a linear system $\dfrac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$, how do we calculate the vector in the vector field at any given point $\mathbf{Y}_0$?

How do we calculate the equilibrium points of $\dfrac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$?

Theorem. The origin is always an equilibrium point of a linear system. It is the only equilibrium point if and only if $\det \mathbf{A} \neq 0$.

The Linearity Principle

Let's return to Example 1. For practice, we'll use vector notation this time:

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \mathbf{Y}$$

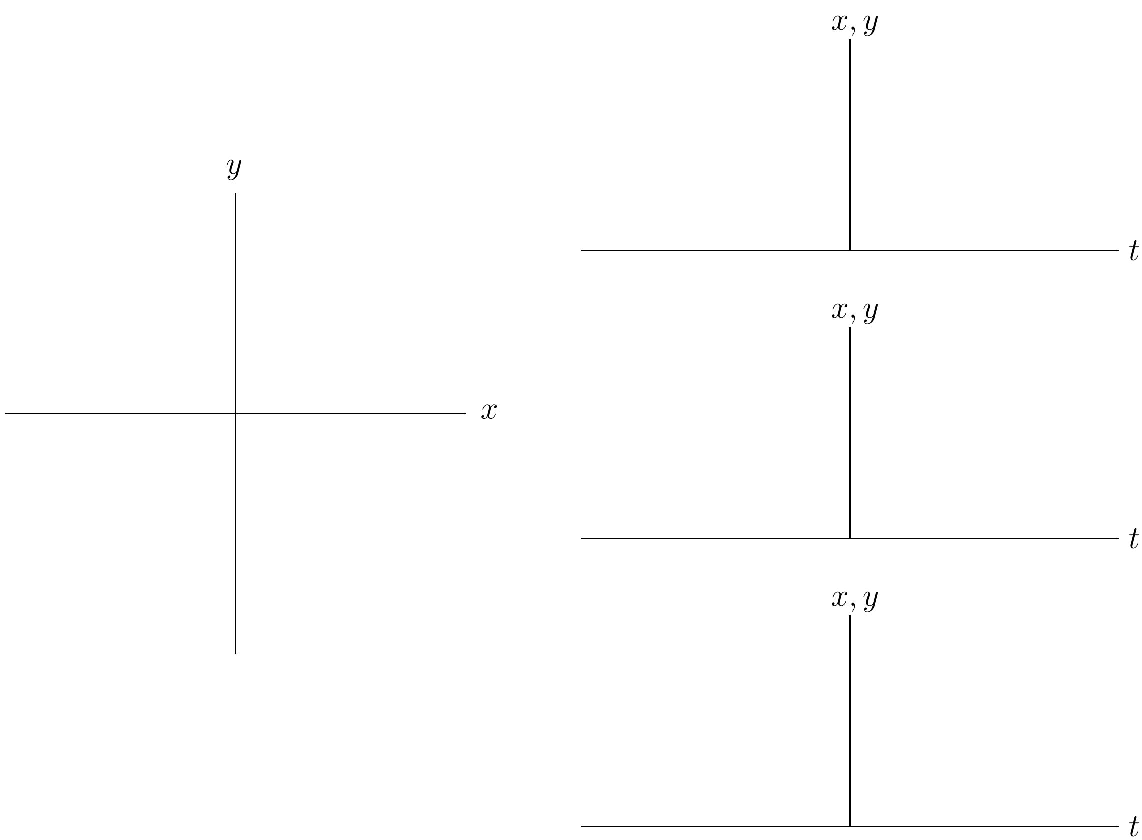
Also consider three different initial conditions

$$\mathbf{Y}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \mathbf{Y}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \mathbf{Y}_3 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

They correspond to the three solutions

$$\mathbf{Y}_1(t) = e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{Y}_2(t) = e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{Y}_3(t) = \begin{pmatrix} e^t + e^{-t} \\ e^t \end{pmatrix}.$$

Let's see what happens when we graph these solutions.



How are these three solutions related?

Linearity Principle Suppose

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$$

is a linear system of differential equations.

1. If $\mathbf{Y}(t)$ is a solution of this system and k is any constant, then $k\mathbf{Y}(t)$ is also a solution.
2. If $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are two solutions of this system, then $\mathbf{Y}_1(t)+\mathbf{Y}_2(t)$ is also a solution.

Example. Solve

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}\mathbf{Y}, \quad \mathbf{Y}(0) = \begin{pmatrix} -1 \\ -2 \end{pmatrix}.$$

For an arbitrary linear system $d\mathbf{Y}/dt = \mathbf{A}\mathbf{Y}$, how many solutions do we need to solve every initial-value problem?