

## Straight-line solutions

Last class we solved the initial-value problem

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \mathbf{Y}, \quad \mathbf{Y}(0) = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

using a linear combination of the two solutions

$$\mathbf{Y}_1(t) = e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We can solve any initial-value problem for this differential equation using an appropriate linear combination of  $\mathbf{Y}_1(t)$  and  $\mathbf{Y}_2(t)$ . In other words, the general solution of this system is

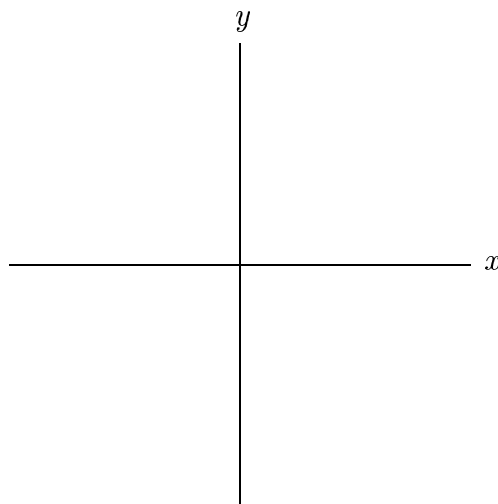
$$\mathbf{Y}(t) = k_1 e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Note the difference between this version of the general solution and the general solution

$$\mathbf{Y}(t) = e^{-t} \begin{pmatrix} x_0 - y_0 \\ 0 \end{pmatrix} + e^t \begin{pmatrix} y_0 \\ y_0 \end{pmatrix},$$

which we obtained by solving this system as a partially decoupled system.

For a general linear system of the form  $d\mathbf{Y}/dt = \mathbf{A}\mathbf{Y}$ , what geometric property of the vector field guarantees the existence of these “straight-line” solutions?



**“Straight-line” Solutions.** Suppose that

$$\mathbf{A}\mathbf{Y}_0 = \lambda\mathbf{Y}_0$$

for some nonzero vector  $\mathbf{Y}_0$  and some scalar  $\lambda$ . Then the function  $\mathbf{Y}(t) = e^{\lambda t}\mathbf{Y}_0$  is a solution to the linear differential equation  $d\mathbf{Y}/dt = \mathbf{A}\mathbf{Y}$ .

We want nonzero initial conditions  $\mathbf{Y}_0$  (vectors) so that

$$\mathbf{A}\mathbf{Y}_0 = \lambda\mathbf{Y}_0$$

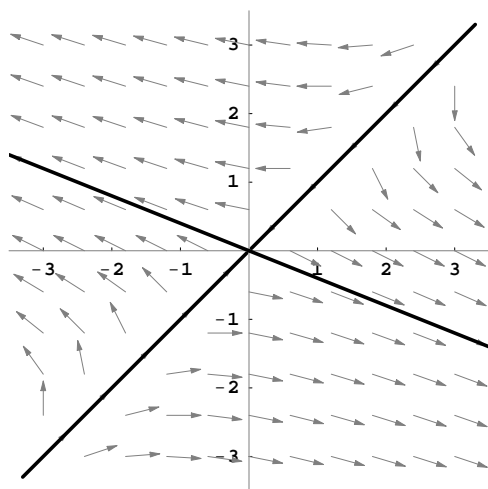
for some scalar  $\lambda$ .

**Terminology:** The scalar  $\lambda$  is called an *eigenvalue* of the matrix  $\mathbf{A}$  and the vector  $\mathbf{Y}_0$  is called an *eigenvector* associated to the eigenvalue  $\lambda$ .

**Example.** Consider

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 4 & -5 \\ -2 & 1 \end{pmatrix} \mathbf{Y}.$$

First let's see what **MatrixFields** tells us about the eigenvalues and eigenvectors of the matrix **A**.



Aside from the theory of algebraic linear equations

For what matrices  $\mathbf{B}$  does the equation  $\mathbf{B}\mathbf{Y} = \mathbf{0}$  have nontrivial solutions?

**Singular Matrices.** The matrix equation  $\mathbf{B}\mathbf{Y} = \mathbf{0}$  has nontrivial solutions  $\mathbf{Y}$  if and only if  $\det \mathbf{B} = 0$ .

**Notes:**

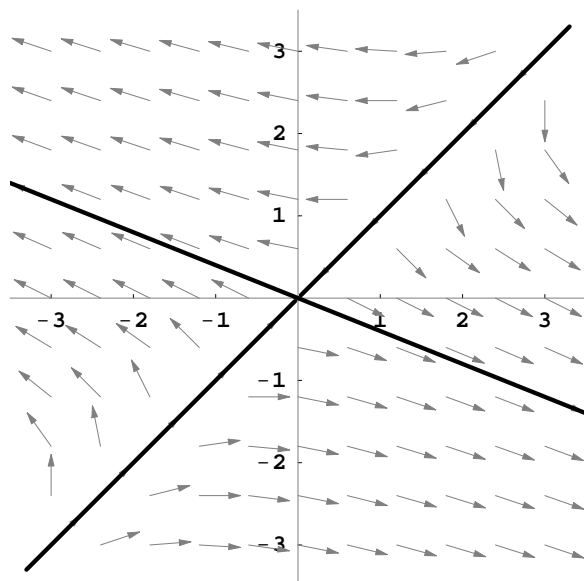
1. Most matrices are nonsingular (not singular).
2. We encountered a singular matrix last class when we studied the linear system that had a line of equilibrium points.

Finding eigenvalues and eigenvectors:

**Example.** Find the general solution to

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 4 & -5 \\ -2 & 1 \end{pmatrix} \mathbf{Y}.$$

Using `HPGSystemSolver`, we plot the phase portrait for this system.



Facts about eigenvalues and eigenvectors: Given a  $2 \times 2$  matrix  $\mathbf{A}$ ,

1. The characteristic equation can have two real roots, one real root of multiplicity two, or two complex conjugate roots.
2. Given an eigenvector  $\mathbf{Y}_0$  associated to an eigenvalue  $\lambda$ , then any nonzero scalar multiple  $\mathbf{Y}_0$  is also an eigenvector associated to  $\lambda$ .
3. Eigenvectors associated to distinct eigenvalues are linearly independent.