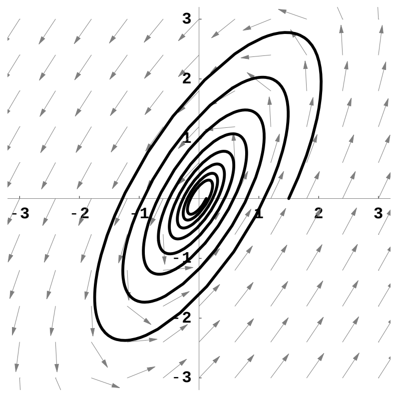
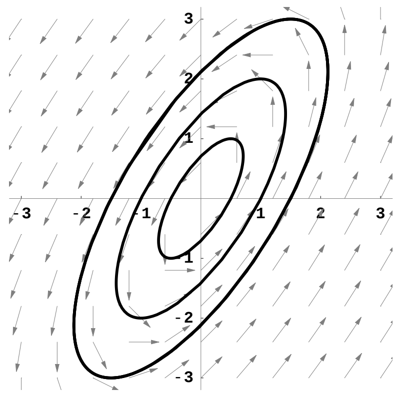
Summary: Linear systems with complex eigenvalues $\lambda = a \pm bi$

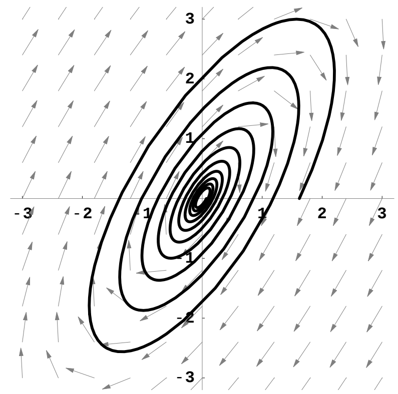
Here are the possible phase portraits:



spiral sink ($a < 0$)



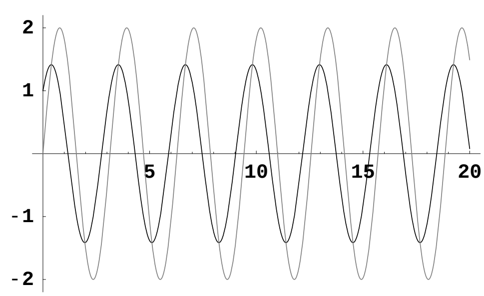
center ($a = 0$)



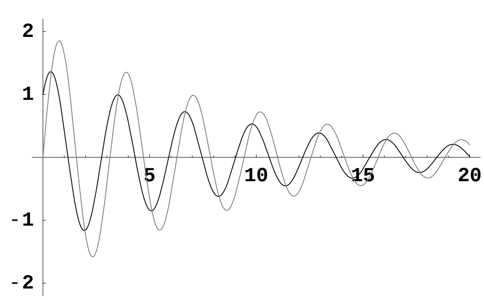
spiral source ($a > 0$)

What information can you get just from the complex eigenvalue alone?

Recall Example 2. The eigenvalues are $\lambda = \pm 2i$. Here are the $x(t)$- and $y(t)$-graphs of a typical solution:



In Example 3, the eigenvalues are $\lambda = -0.1 \pm 2i$. Here are the $x(t)$- and $y(t)$-graphs of a typical solution:



Frequency versus period: The solutions in Example 3 are not periodic in the strict sense. There is no time T such that

$$x(t+T) = x(t) \quad \text{and} \quad y(t+T) = y(t)$$

for all t. However, there is a period associated to these solutions. In the text, we call this the **natural period** of the solutions.

Perhaps it is best to think about these solutions as oscillating solutions that are decaying over time and to measure the oscillations in terms of their **frequency**.

Definition. The *frequency* F of an oscillating function $g(t)$ is the number of cycles that $g(t)$ makes in one unit of time.

Suppose that $g(t)$ is oscillating periodically with "period" T. What is its frequency F?

Example. Consider the standard sinusoidal functions $g(t) = \cos \beta t$ and $g(t) = \sin \beta t$.

Suppose we measure frequency in radians rather than in cycles. This measure of frequency is often called **angular frequency**. Let's denote the angular frequency by f. Then

$$f = 2\pi F.$$

Repeated eigenvalues

Sometimes the characteristic polynomial has the same real root twice. When this happens, we say that the eigenvalues are "repeated."

Example. $\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$ where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -4 & -4 \end{pmatrix}.$$

The characteristic polynomial of $\mathbf{A}$ is $\lambda^2 + 4\lambda + 4$, so $\lambda = -2$ is a repeated eigenvalue. Let's calculate all straight-line solutions.

We will find the general solution in this case using an entirely different approach. Recall that the solution to the first-order linear initial-value problem

$$\frac{dy}{dt} = ay, \quad y(0) = y_0$$

is $y(t) = y_0 e^{at}$.

We can do the same thing with linear systems. Given the initial-value problem

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}, \quad \mathbf{Y}(0) = \mathbf{Y}_0,$$

we can obtain a solution of the form $\mathbf{Y}(t) = e^{t\mathbf{A}}\mathbf{Y}_0$ once we understand what $e^{t\mathbf{A}}$ is.

Definition. The function $e^{t\mathbf{A}}$ is the matrix-valued function defined by

$$e^{t\mathbf{A}} = \sum_{k=0}^{\infty} \frac{(t\mathbf{A})^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathbf{A}^k.$$

To understand the definition, we do two examples.

Example. Let $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

Example. Let $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$.

Let's suppose that the definition of $e^{t\mathbf{A}}$ makes sense. Then

$$\frac{d}{dt}\,e^{t\mathbf{A}} =$$

Note also that $e^{t\mathbf{A}}\Big|_{t=0} = \mathbf{I}$.

Using these two facts, we see that the function $\mathbf{Y}(t) = e^{t\mathbf{A}}\mathbf{Y}_0$ solves the initial-value problem

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}, \quad \mathbf{Y}(0) = \mathbf{Y}_0.$$

Unfortunately it is not so easy to compute $e^{t\mathbf{A}}$ in general.

An eigenvector $\mathbf{Y}_0$ associated to an eigenvalue λ is a vector that satisfies the equation

$$\mathbf{A}\mathbf{Y}_0 = \lambda\mathbf{Y}_0.$$

This equation can be rewritten as $(\mathbf{A} - \lambda\mathbf{I})\mathbf{Y}_0 = \mathbf{0}$. A *generalized eigenvector* $\mathbf{Y}_0$ is a vector that satisfies the equation

$$(\mathbf{A} - \lambda\mathbf{I})^k\mathbf{Y}_0 = \mathbf{0}$$

for some $k \geq 2$.

Example. Once again consider the matrix $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -4 & -4 \end{pmatrix}$.

To see why generalized eigenvectors are well suited to the matrix exponential, let's suppose that $(\mathbf{A} - \lambda\mathbf{I})^2\mathbf{Y}_0 = \mathbf{0}$ and compute

$$\mathbf{Y}(t) = e^{t\mathbf{A}}\mathbf{Y}_0$$

by writing $\mathbf{A} = \lambda\mathbf{I} + (\mathbf{A} - \lambda\mathbf{I})$.