Summary: Linear systems with complex eigenvalues $\lambda = a \pm bi$

Here are the possible phase portraits:

- **spiral sink** ($a < 0$)
- **center** ($a = 0$)
- **spiral source** ($a > 0$)

What information can you get just from the complex eigenvalue alone?

Recall Example 2. The eigenvalues are $\lambda = \pm 2i$. Here are the $x(t)$- and $y(t)$-graphs of a typical solution:

In Example 3, the eigenvalues are $\lambda = -0.1 \pm 2i$. Here are the $x(t)$- and $y(t)$-graphs of a typical solution:
Frequency versus period: The solutions in Example 3 are not periodic in the strict sense. There is no time \( T \) such that
\[
x(t + T) = x(t) \quad \text{and} \quad y(t + T) = y(t)
\]
for all \( t \). However, there is a period associated to these solutions. In the text, we call this the **natural period** of the solutions.

Perhaps it is best to think about these solutions as oscillating solutions that are decaying over time and to measure the oscillations in terms of their **frequency**.

**Definition.** The frequency \( F \) of an oscillating function \( g(t) \) is the number of cycles that \( g(t) \) makes in one unit of time.

Suppose that \( g(t) \) is oscillating periodically with “period” \( T \). What is its frequency \( F \)?

**Example.** Consider the standard sinusoidal functions \( g(t) = \cos \beta t \) and \( g(t) = \sin \beta t \).

Suppose we measure frequency in radians rather than in cycles. This measure of frequency is often called **angular frequency**. Let’s denote the angular frequency by \( f \). Then
\[
f = 2\pi F.
\]
Repeated eigenvalues

Sometimes the characteristic polynomial has the same real root twice. When this happens, we say that the eigenvalues are “repeated.”

Example. \( \frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y} \) where

\[
\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -4 & -4 \end{pmatrix}.
\]

The characteristic polynomial of \( \mathbf{A} \) is \( \lambda^2 + 4\lambda + 4 \), so \( \lambda = -2 \) is a repeated eigenvalue. Let’s calculate all straight-line solutions.

We will find the general solution in this case using an entirely different approach. Recall that the solution to the first-order linear initial-value problem

\[
\frac{dy}{dt} = ay, \quad y(0) = y_0
\]

is \( y(t) = y_0e^{at} \).

We can do the same thing with linear systems. Given the initial-value problem

\[
\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}, \quad \mathbf{Y}(0) = \mathbf{Y}_0,
\]

we can obtain a solution of the form \( \mathbf{Y}(t) = e^{t\mathbf{A}}\mathbf{Y}_0 \) once we understand what \( e^{t\mathbf{A}} \) is.
Definition. The function $e^{tA}$ is the matrix-valued function defined by

$$e^{tA} = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k.$$ 

To understand the definition, we do two examples.

Example. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

Example. Let $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. 
Let’s suppose that the definition of $e^{tA}$ makes sense. Then

$$\frac{d}{dt} e^{tA} =$$

Note also that $e^{tA}|_{t=0} = I$.

Using these two facts, we see that the function $Y(t) = e^{tA}Y_0$ solves the initial-value problem

$$\frac{dY}{dt} = AY, \quad Y(0) = Y_0.$$ 

Unfortunately it is not so easy to compute $e^{tA}$ in general.

An eigenvector $Y_0$ associated to an eigenvalue $\lambda$ is a vector that satisfies the equation

$$AY_0 = \lambda Y_0.$$ 

This equation can be rewritten as $(A - \lambda I)Y_0 = 0$. A generalized eigenvector $Y_0$ is a vector that satisfies the equation

$$(A - \lambda I)^k Y_0 = 0$$

for some $k \geq 2$. 

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Example. Once again consider the matrix $A = \begin{pmatrix} 0 & 1 \\ -4 & -4 \end{pmatrix}$.

To see why generalized eigenvectors are well suited to the matrix exponential, let’s suppose that $(A - \lambda I)^2 Y_0 = 0$ and compute

$$Y(t) = e^{tA}Y_0$$

by writing $A = \lambda I + (A - \lambda I)$. 