General second-order autonomous equations

In general, a second-order autonomous equation has

- one independent variable and
- one dependent variable.

It has the form $\dfrac{d^2y}{dt^2} = f\left(y, \dfrac{dy}{dt}\right)$.

Example. Simple mass-spring system

Hooke's Law: The restoring force of the spring is proportional to the displacement from its rest position.

Using Newton's law $F = ma$, we get

Let's consider the special case where $k = m$. We get $\dfrac{d^2y}{dt^2} = -y$, and we can guess some solutions to this equation:

General 2D first-order autonomous systems

In general, a 2D first-order autonomous system of ordinary differential equations has

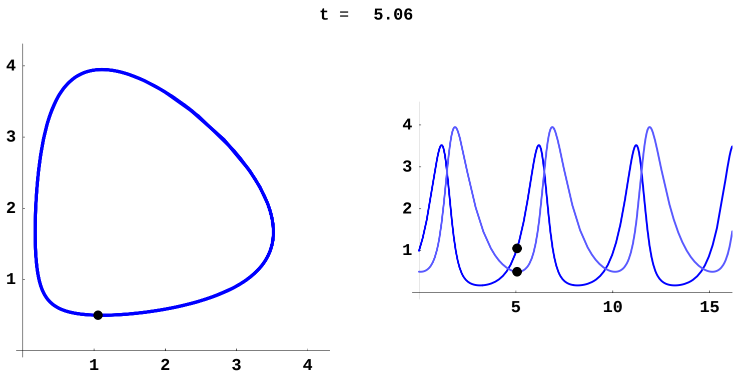
- one independent variable and
- two dependent variables.
- The independent variable does not appear on the right-hand sides of the differential equations.

Example. Recall the predator-prey systems we discussed briefly at the start of the semester

$$\frac{dR}{dt} = aR - bRF$$
$$\frac{dF}{dt} = -cF + dRF.$$

Let's go through some terminology:

- initial condition:
- solution to an initial-value problem:



The solution shown above corresponds to the initial condition $(R_0, F_0) = (1, 0.5)$ with parameter values $a = 2$, $b = 1.2$, $c = 1$, and $d = 0.9$. See the web site for the entire animation and for a related 3D animation. `DETools` also has a tool called `PredatorPrey`.

- component graphs:
- phase plane:
- solution curve in the phase plane:
- equilibrium solutions:

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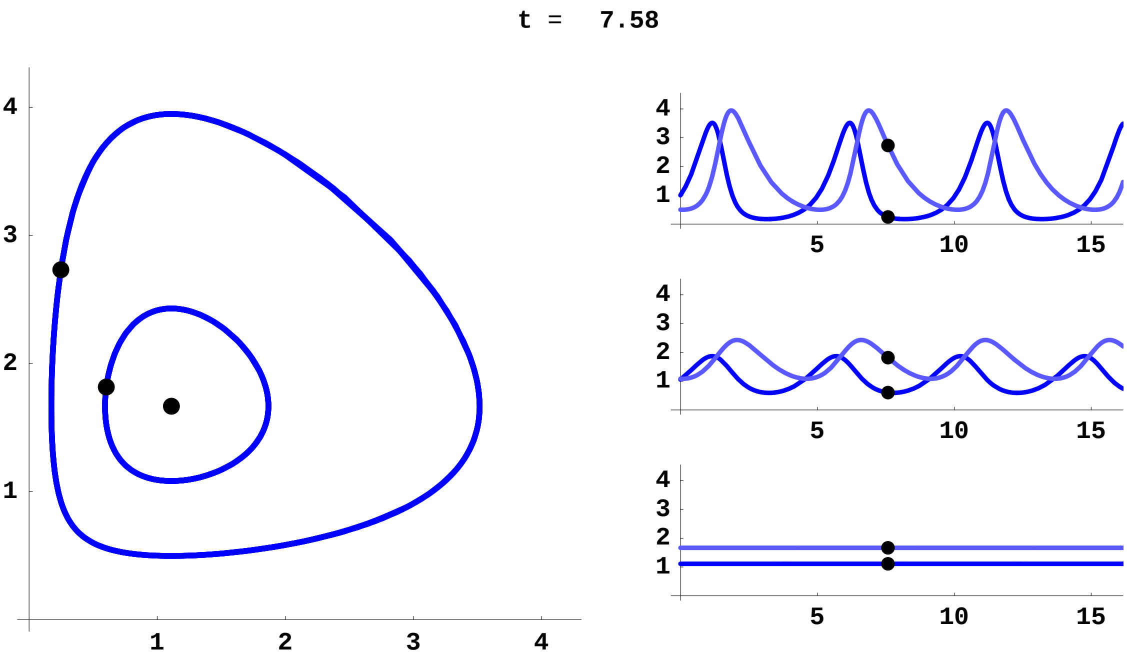

- phase portrait:

One skill that you will learn is how to make a rough sketch of the component graphs from the solution curve. There is a tool on your CD called `DESketchPad` which will help you practice.

Although these two examples seem quite different, they have more in common than one might think. In particular, the mass-spring system can also be written as a first-order system by introducing the "new" variable v (which is just dy/dt). We get

In what ways is the mass-spring system similar to the predator-prey system?

An initial condition for the predator-prey system is a pair (R_0, F_0) of population values.

An initial condition for the mass-spring system is also a pair (y_0, v_0). The first number indicates the initial displacement and the second number indicates the initial velocity.

Since the mass-spring system can be expressed as a system, all of the terms that we discussed for the predator-prey system apply to the mass-spring system as well (equilibrium solutions, component graphs, phase plane, solution curve, phase portrait, ...). There are two animations on the class web site that illustrate these ideas for the mass-spring system.

One way that the two systems differ is by the fact that we can find formulas for the solutions of the mass-spring system but not for the predator-prey system. In fact, when we considered the special case of the mass-spring system with $k = m$, we got

$$\frac{d^2y}{dt^2} = -y,$$

and we guessed the solutions

$$\begin{aligned} y_1(t) &= \sin t, \\ y_2(t) &= \cos t, \text{ and} \\ y_3(t) &= 2\sin t. \end{aligned}$$

The equivalent system

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -y$$

has one $(y(t), v(t))$ pair of solutions for each solution to the second-order equation:

What are the initial conditions for these solutions?

Where do we go from here?

1. Vector fields (similar to slope fields)
2. Two analytic techniques (more guessing)
3. Euler's method again
4. Some theory (Existence and Uniqueness)
5. Linear systems and equations (Chapter 3)