

The vector field of an autonomous system

We get a better geometric understanding of the solutions of a first-order system of differential equations if we rewrite the system as a vector equation that applies to a vector-valued function.

Consider the system

$$\begin{aligned}\frac{dx}{dt} &= f(x, y) \\ \frac{dy}{dt} &= g(x, y)\end{aligned}$$

with independent variable t and dependent variables x and y . We use the right-hand side of this system to form a vector field

$$\mathbf{F} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix}$$

in the xy -plane. We also use $x(t)$ and $y(t)$ to form a vector-valued function

$$\mathbf{Y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Then the (scalar) system of differential equations can be rewritten as one vector differential equation

$$\frac{d\mathbf{Y}}{dt} = \mathbf{F}(\mathbf{Y}).$$

Example 1. We consider the simple mass-spring system

$$\begin{aligned}\frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -y.\end{aligned}$$

First, let's rewrite this system in vector notation:

Consider the solution $(y_2(t), v_2(t)) = (\cos t, -\sin t)$ from last class. Let's express it in vector notation:

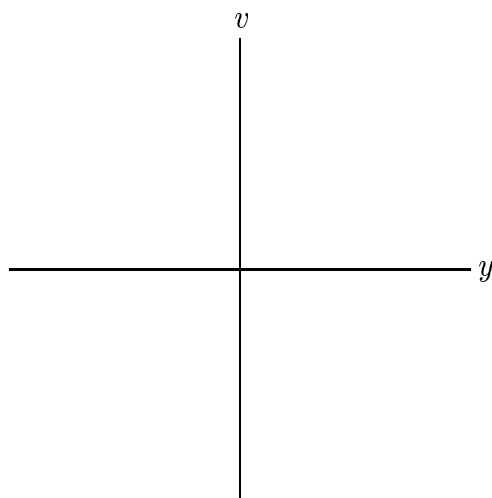
Now for the geometric interpretation of

$$\frac{d\mathbf{Y}}{dt} = \mathbf{F}(\mathbf{Y}),$$

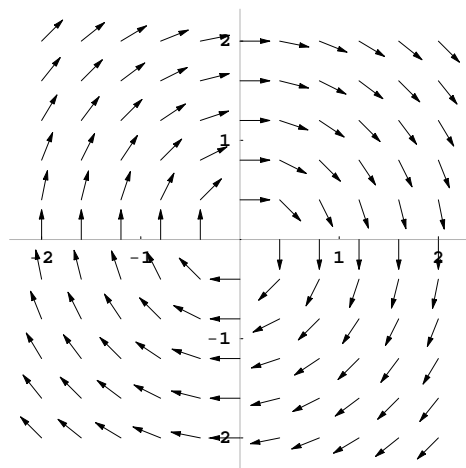
where

$$\mathbf{F}(\mathbf{Y}) = \mathbf{F} \begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} v \\ -y \end{pmatrix}.$$

We use `HPGSystemSolver` to help visualize the vector field and the solutions.



The direction field associated with this system is



Here's another example:

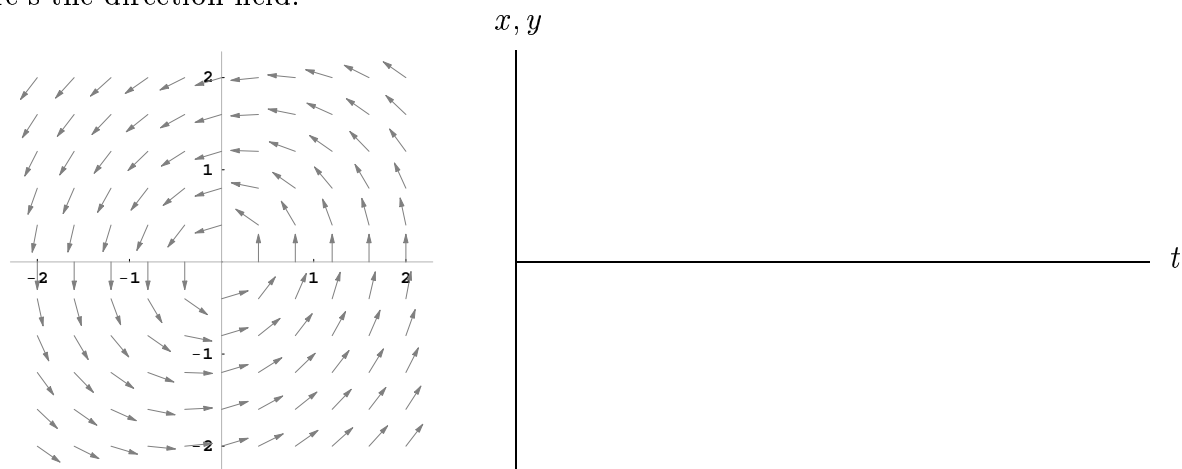
Example 2. Consider the system

$$\begin{aligned}\frac{dx}{dt} &= -y \\ \frac{dy}{dt} &= x - 0.3y.\end{aligned}$$

The vector field associated with this system is

$$\mathbf{F}(\mathbf{Y}) = \mathbf{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x - 0.3y \end{pmatrix}.$$

Here's the direction field:



In this week's homework you have a matching problem in which you match systems of equations with their corresponding direction fields. There is another one of these problems in the old exams. Doing these problems is a good way to make sure that you understand how a system of differential equations determines a vector field.

Analytic techniques:

There are few analytic techniques that work for both linear and nonlinear systems.

1. You can always check to see if a given function is a solution (no wrong answers).

For example, consider the initial-value problem

$$\begin{aligned} \frac{dx}{dt} &= 2y - x \\ \frac{dy}{dt} &= y \end{aligned} \quad (x_0, y_0) = (2, 1).$$

Using the vector notation $\mathbf{Y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ we can write this IVP as

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2y - x \\ y \end{pmatrix}, \quad \mathbf{Y}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

First, let's see what the solution looks like when we graph it with `HPGSystemSolver`:



Claim: The function $\mathbf{Y}(t) = \begin{pmatrix} e^t + e^{-t} \\ e^t \end{pmatrix}$ solves the IVP.

2. General solution of a partially-decoupled system

Example. Consider the previous system

$$\begin{aligned}\frac{dx}{dt} &= 2y - x \\ \frac{dy}{dt} &= y.\end{aligned}$$

We can calculate the general solution using methods we learned for first-order equations:

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Damped harmonic oscillator

Let's return to our mass-spring system and add a term that models damping.

Assumption: The damping force is proportional to the speed of the mass and it acts as a restoring force.

The second-order equation

$$m \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + ky = 0$$

and its equivalent system

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -\frac{k}{m}y - \frac{b}{m}v \end{aligned}$$

appear in many applications. On the CD, you will find it in `MassSpring` and `RLCCircuits`, and it has also been used to study biological processes such as the blood glucose regulatory system in humans.

There is a guessing technique for the damped harmonic oscillator

$$m \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + ky = 0.$$

Example. Consider the harmonic oscillator

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0.$$

Its characteristic equation is

Let's plot these solutions with `HPGSystemSolver`. What are the corresponding solution curves and component graphs?

