More on the matrix exponential and repeated eigenvalues

Last class we saw that the function $\mathbf{Y}(t) = e^{t\mathbf{A}}\mathbf{Y}_0$ solves the initial-value problem

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}, \quad \mathbf{Y}(0) = \mathbf{Y}_0.$$

Unfortunately it is not so easy to compute $e^{t\mathbf{A}}$ in general.

An eigenvector $\mathbf{Y}_0$ associated to an eigenvalue λ is a vector that satisfies the equation

$$\mathbf{A}\mathbf{Y}_0 = \lambda\mathbf{Y}_0.$$

This equation can be rewritten as $(\mathbf{A} - \lambda\mathbf{I})\mathbf{Y}_0 = \mathbf{0}$. A *generalized eigenvector* $\mathbf{Y}_0$ is a vector that satisfies the equation

$$(\mathbf{A} - \lambda\mathbf{I})^k\mathbf{Y}_0 = \mathbf{0}$$

for some $k \geq 2$.

Example. Once again consider the matrix $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -4 & -4 \end{pmatrix}$.

To see why generalized eigenvectors are well suited to the matrix exponential, let's suppose that $(\mathbf{A} - \lambda\mathbf{I})^2\mathbf{Y}_0 = \mathbf{0}$ and compute

$$\mathbf{Y}(t) = e^{t\mathbf{A}}\mathbf{Y}_0$$

by writing $\mathbf{A} = \lambda\mathbf{I} + (\mathbf{A} - \lambda\mathbf{I})$.

Fact from linear algebra: If $\mathbf{A}$ is a 2×2 matrix with a repeated eigenvalue λ and $\mathbf{V}_0$ is any vector, then either

1. $(\mathbf{A} - \lambda\mathbf{I})\mathbf{V}_0 = \mathbf{0}$ (in other words, $\mathbf{V}_0$ is an eigenvector), or
2. the vector $\mathbf{V}_1 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{V}_0$ is an eigenvector of $\mathbf{A}$.

Example. Yet again consider $\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$ where

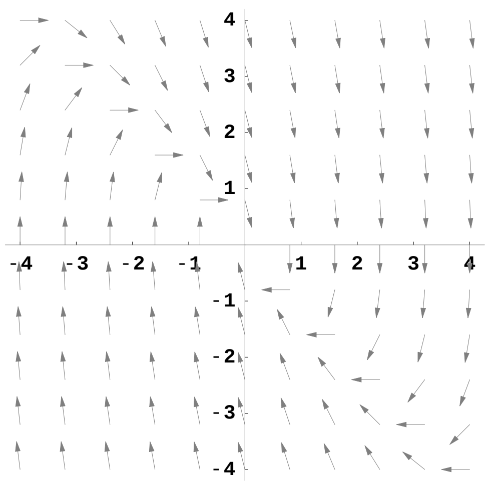
$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -4 & -4 \end{pmatrix}.$$

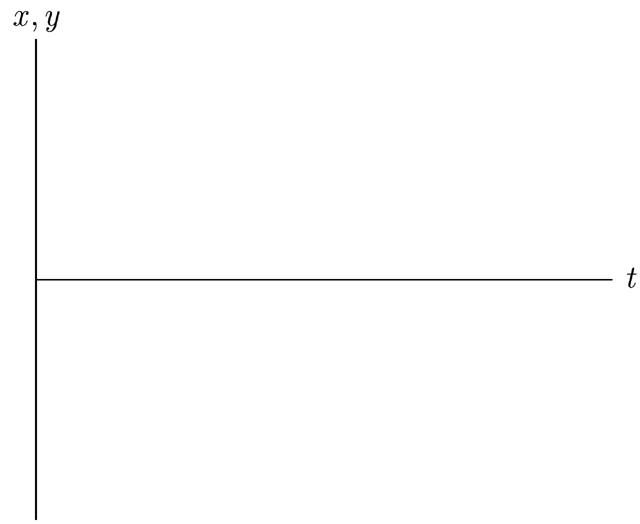
Recall that $\lambda = -2$ is a repeated eigenvalue.

What is the long-term behavior of a system with a repeated, negative eigenvalue?

It is interesting to look at this example using two of the tools on the CD. Using `LinearPhasePortraits`, we can see that this system is on the boundary between spiral sinks and real sinks.

We can also use `HPGSystemSolver` to plot the phase portrait and a typical pair of $x(t)$- and $y(t)$-graphs.



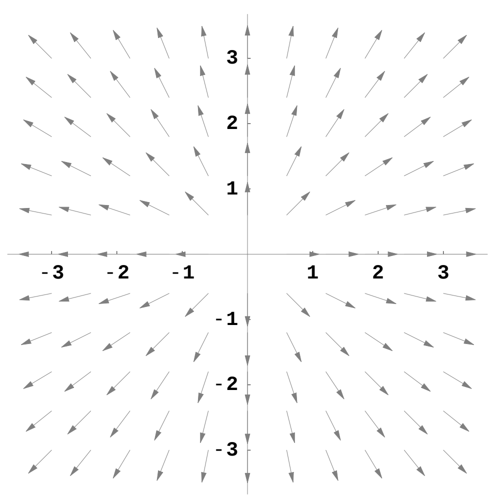


Unusual case of repeated eigenvalues: There is one type of linear system that has repeated eigenvalues that is different from the examples we have discussed.

Example. Consider $d\mathbf{Y}/dt = \mathbf{AY}$ where $\mathbf{A}$ is the diagonal matrix

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

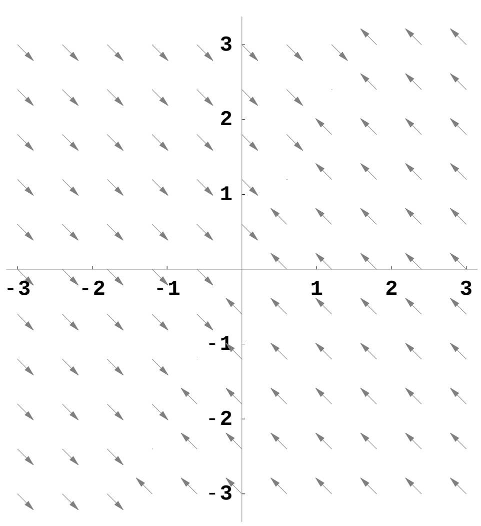
What are its eigenvalues and eigenvectors?

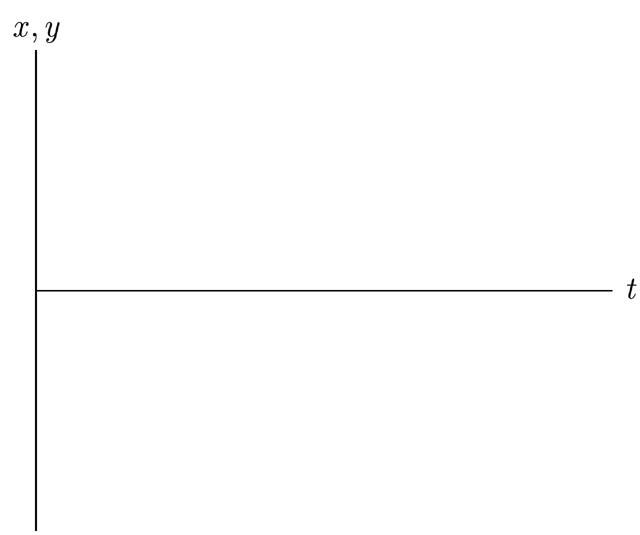


Finally consider the example

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix} \mathbf{Y}.$$

Its characteristic polynomial is $\lambda^2 + 3\lambda$. So its eigenvalues are $\lambda = -3$ and $\lambda = 0$. (If a system has 0 as an eigenvalue, we say that it is *degenerate*. The matrix $\mathbf{A}$ of coefficients is singular—see your class notes for October 20.)





Second-order, linear equations

We now apply what we have learned about linear systems to solve second-order homogeneous linear equations.

Let's return to the guessing technique for second-order equations that we learned about a month ago (see Section 2.3 in the text and your class notes from October 8 and 15). In particular, let's see how it relates to what we have done with linear systems recently.

Example. Consider the equation

$$2\frac{d^2y}{dt^2} + 9\frac{dy}{dt} + 4y = 0.$$

1. Use a guessing technique to find two nonzero solutions $y_1(t)$ and $y_2(t)$ that are not multiples of each other.

2. Convert this equation to a first-order system and determine the analogous solutions $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$.

3. In what way are $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ special solutions?

Let's see how this guessing technique can be used to solve all second-order homogeneous equations.

Consider

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0$$

with its characteristic equation

$$a\lambda^2 + b\lambda + c = 0$$

as well as the corresponding system

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -\frac{c}{a}\,y - \frac{b}{a}\,v \end{aligned}$$

with its characteristic equation

$$\det\begin{pmatrix} -\lambda & 1 \\ -\dfrac{c}{a} & -\dfrac{b}{a} - \lambda \end{pmatrix} = 0.$$

Useful observation: If λ is an eigenvalue, the vector

$$\mathbf{Y}_0 = \begin{pmatrix} 1 \\ \lambda \end{pmatrix}$$

is *always* an associated eigenvector.

Let's see what that observation tells us about solutions to the second-order equation. There are three cases:

1. Two real, distinct, nonzero eigenvalues λ_1 and λ_2:

2. A complex-conjugate pair of eigenvalues $\lambda = \alpha \pm i\beta$, with $\beta \neq 0$:

3. One nonzero real eigenvalue λ of multiplicity two:

Conclusion: We can determine the general solution of a homogeneous linear second-order equation

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0$$

immediately from the characteristic equation $a\lambda^2 + b\lambda + c = 0$.

YOU DO NOT NEED TO CALCULATE THE EIGENVECTORS OR EVEN REDUCE TO A FIRST-ORDER SYSTEM if you simply want to produce the general solution of a linear second-order equation.

Example. Let's compute the general solution to

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0.$$