A little more about linear systems/equations

We can apply what we have learned about homogeneous second-order equations to the (damped) harmonic oscillator
$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0.$$
In this case, we are assuming that the parameters m and k are positive and that $b \geq 0$. The characteristic equation $m\lambda^2 + b\lambda + k = 0$ has eigenvalues
$$\frac{-b \pm \sqrt{b^2 - 4mk}}{2m}.$$

There are three cases based on the value of the discriminant $b^2 - 4mk$.

1. $b^2 - 4mk < 0$: In this case, the eigenvalues are complex and can be written as
$$\lambda = \left(-\frac{b}{2m}\right) \pm \left(\frac{\sqrt{4mk - b^2}}{2m}\right) i.$$
The real part determines the exponential decay rate for solutions, and the imaginary part determines the natural "period" of the solutions.

   There are two subcases:

   (a) $b = 0$: all solutions are periodic. This is the **undamped** case.

   (b) $b \neq 0$: solutions oscillate with a constant frequency, but they decay at an exponential rate. This is the **underdamped** case.

2. $b^2 - 4mk = 0$: The eigenvalue
$$\lambda = -\frac{b}{2m}$$
is repeated. This is the **critically damped** case. In this case, solutions approach zero as rapidly as possible.

3. $b^2 - 4mk > 0$: The eigenvalues
$$\lambda = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$
are both real. Note that
$$0 < b^2 - 4mk < b^2.$$
Therefore, both eigenvalues are negative, and the equilbrium point at the origin is a (real) sink. The rate of approach to zero by a typical solution is determined by the "slow" eigenvalue. This is the **overdamped** case.

Example. Consider the one-parameter family of equations

$$\frac{d^2y}{dt^2} + b\frac{dy}{dt} + y = 0.$$

In this case, the characteristic equation is $\lambda^2 + b\lambda + 1 = 0$, and consequently, the eigenvalues are

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4}}{2}.$$

The value $b = 2$ is the critical value for this family.

We can see the progression from underdamped to critically damped to overdamped with a Quicktime animation that I have posted on the web site.

The trace-determinant plane

There is a nice geometric object called the trace-determinant plane that organizes the various types of 2×2 linear systems.

Consider the 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let's calculate the characteristic polynomial of $\mathbf{A}$:

Conclusion: The eigenvalues of any 2×2 matrix are determined by the trace and the determinant of $\mathbf{A}$. We have

$$\lambda = \frac{(\mathrm{tr}\mathbf{A}) \pm \sqrt{(\mathrm{tr}\mathbf{A})^2 - 4(\det \mathbf{A})}}{2}.$$

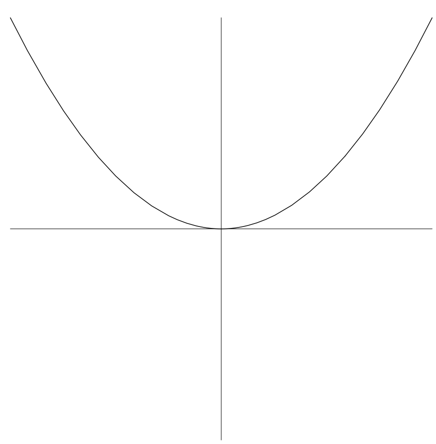
Summary of Phase Portraits

Assume $\det \mathbf{A} \neq 0$. Then zero is not an eigenvalue of $\mathbf{A}$.

1. Real and distinct eigenvalues
   (a) sink
   (b) saddle
   (c) source
2. Complex eigenvalues
   (a) spiral sink
   (b) center
   (c) spiral source
3. Real and repeated eigenvalues
   (a) sink with one eigenline in the phase portrait
   (b) source with one eigenline in the phase portrait
   (c) sink where every solution is a straight-line solution
   (d) source where every solution is a straight-line solution

What if $\det \mathbf{A} = 0$?

We can organize these different types using a plane with unusual coordinate axes.



You can turn on the trace-determinant plane in the `LinearPhasePortraits` tool.

Forced equations

For the last five weeks of the semester, all of our differential equations have been autonomous. Now we turn to second-order equations that model systems that are subject to some type of external forcing. Here are three examples:

Example. The nonlinear pendulum with a pivot point that is subject to vertical oscillations. The motion of such a pendulum is governed by the second-order nonlinear equation

$$m\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + k\sin\theta = F\sin\theta\cos\omega t$$

where ω determines the frequency of the oscillations of the pivot point and F determines the amplitude of the oscillations. The `Pendulums` tool on the CD illustrates this system.

Example. The linear mass-spring system where the spring is subject to vertical oscillations. To model this system, we use the standard mass-spring system and add a term that corresponds to the force added to the system by the oscillations. We get

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = F\cos\omega t.$$

The `ForcedMassSpring` tool on the CD illustrates this system.

Example. The classic RLC circuit is also modeled by a linear, forced second-order equation. On the CD, it is modeled by an equation that involves both charge and current. In our text, we tend to use the equation

$$LC\frac{d^2v_c}{dt^2} + RC\frac{dv_c}{dt} + v_c = V_s(t)$$

where v_c is the voltage across the capacitor and R, L, and C are the resistance, inductance, and capacitance parameters. The forcing term $V_s(t)$ is a voltage source which can change with time. The `RLCCircuits` tool on the CD illustrates this system with a sinusoidal forcing function.

Our success studying unforced linear systems was due in large part to the Linearity Principle. For forced linear equations, we are fortunate to have the Extended Linearity Principle.

Extended Linearity Principle Consider a nonhomogeneous equation (a forced equation)

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = g(t)$$

and its associated homogeneous equation (the unforced equation)

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0.$$

1. Suppose $y_p(t)$ is a particular solution of the nonhomogeneous equation and $y_h(t)$ is a solution of the associated homogeneous equation. Then $y_h(t) + y_p(t)$ is also a solution of the nonhomogeneous equation.

2. Suppose $y_p(t)$ and $y_q(t)$ are two solutions of the nonhomogeneous equation. Then $y_p(t) - y_q(t)$ is a solution of the associated homogeneous equation.

Therefore, if $k_1y_1(t) + k_2y_2(t)$ is the general solution of the associated homogeneous equation, then

$$k_1y_1(t) + k_2y_2(t) + y_p(t)$$

is the general solution of the nonhomogeneous equation.

This principle provides the basic framework that we will use to solve linear second-order forced equations. (At this point in the course, you should go back and review the method described in Section 1.8 for solving nonhomogeneous first-order linear equations.)

We already know how to find the general solution to the associated homogeneous equation, so we need only find one solution to the original equation.

Example 1. Consider the equation

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = e^{3t}.$$

Here's another example that looks similar but goes somewhat differently.

Example 2. Consider the equation

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = e^{-t}.$$

(More blank space on the top of the next page.)

A time saver: There's a calculation that we've already done twice before. It is also useful for guessing $y_p(t)$. Consider the function $y_p(t) = \alpha e^{\lambda t}$ and calculate

$$a\frac{d^2y_p}{dt^2} + b\frac{dy_p}{dt} + cy_p =$$

Let's see how this works in Example 1.

Example 1. Recall

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = e^{3t}.$$