Sinusoidal forcing in the absense of damping

Now consider the mass-spring system without the dashpot.

Example. Let's find the general solution to

$$\frac{d^2y}{dt^2} + 3y = \cos \omega t.$$

Note the lack of a damping term. We want to see what happens with various forcing frequencies.

(Additional blank space on the top of the next page.)

Unfortunately the parts of the solution that correspond to the associated homogeneous equation do not die out. So to get some qualitative understanding in this case, we make a simplifying assumption. We consider the solution that satisfies the initial condition $(y(0), y'(0)) = (0, 0)$.

On the web site, there is a Quicktime animation of the graphs of these solutions as we vary the forcing frequency ω. We can also visualize these solutions using a parameter in `HPGSystemSolver`.

Trig identity:

$$\cos at - \cos bt = -2\left(\sin \alpha t\right)\left(\sin \beta t\right)$$

where

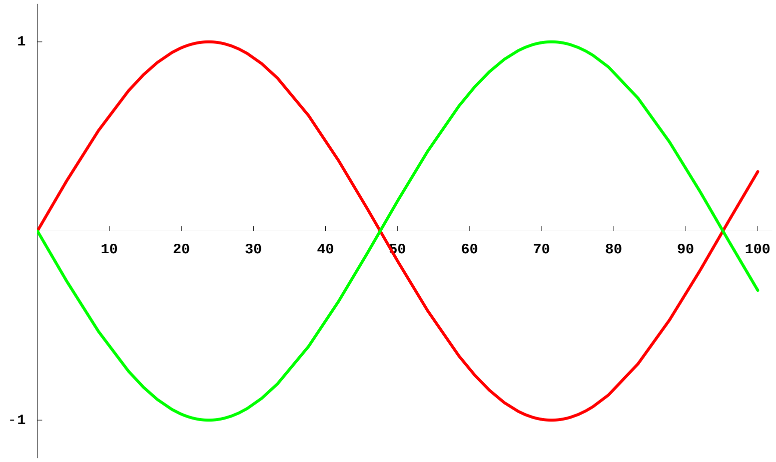
$$\alpha = \frac{a+b}{2} \quad \text{and} \quad \beta = \frac{a-b}{2}.$$

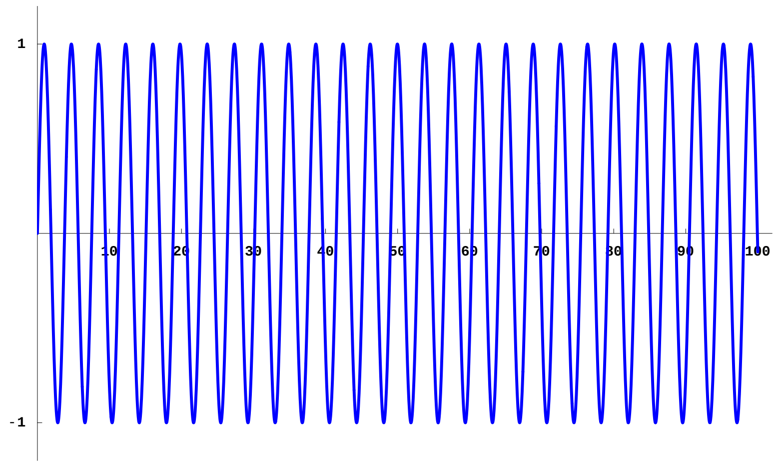
The number α is the average of a and b, and β is called the *half-difference* of a and b.

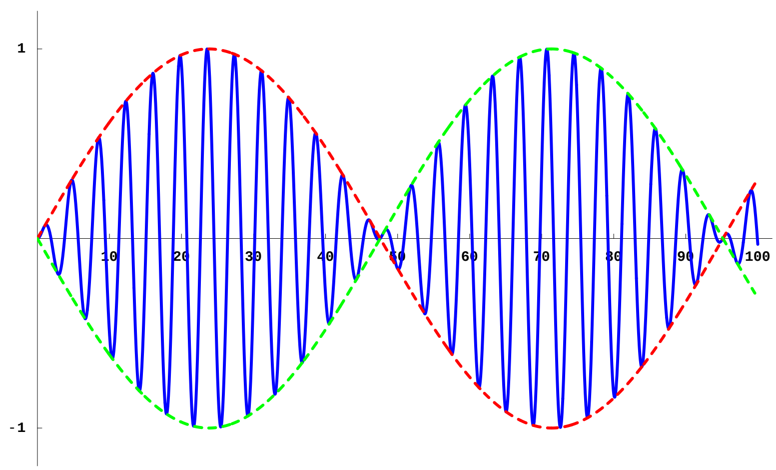
Example. Let's use this trig identity to get a rough idea of the graph of

$$\cos \omega t - \cos \sqrt{3}\, t$$

where $\omega = 1.6$.


1
-1
10
20
30
40
50
60
70
80
90
100


1
-1
10
20
30
40
50
60
70
80
90
100


1
-1
10
20
30
40
50
60
70
80
90
100

Let's return to the solution to

$$\frac{d^2y}{dt^2} + 3y = \cos \omega t$$

that satisfies the initial condition $(y(0), y'(0)) = (0, 0)$. If $\omega \neq \pm\sqrt{3}$, the solution is

$$y(t) = \frac{1}{3 - \omega^2}(\cos \omega t - \cos \sqrt{3}\, t).$$

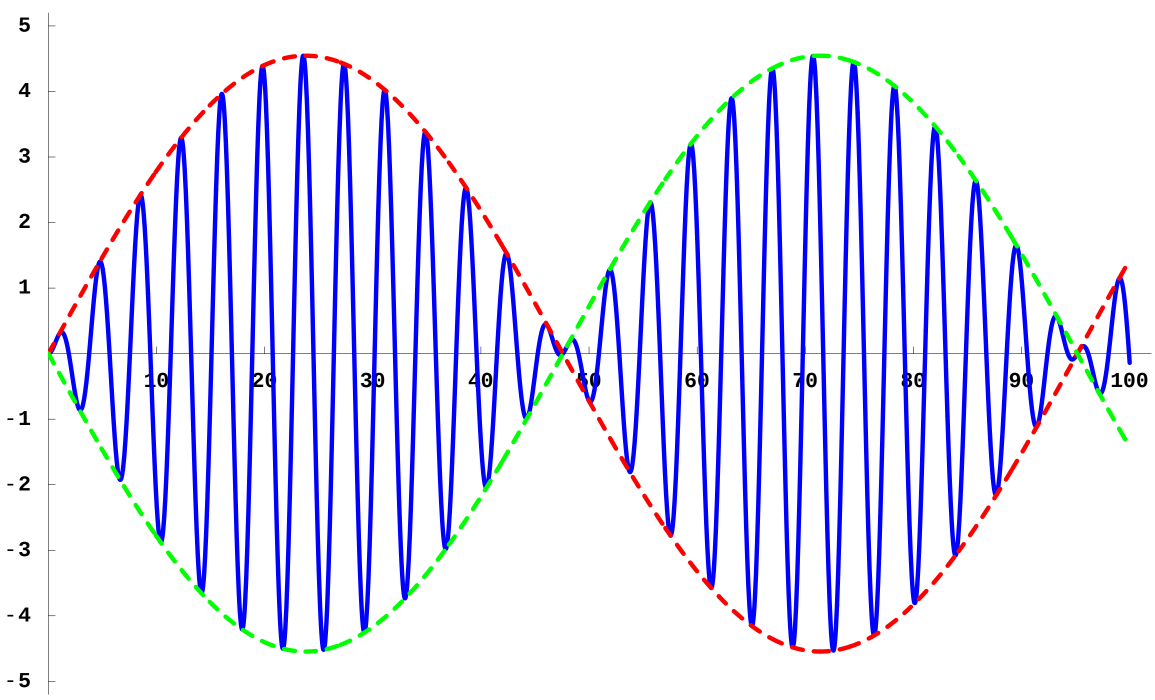
Applying the trig identity, we obtain

$$y(t) = \frac{-2}{3 - \omega^2}\ (\sin \alpha t)\ (\sin \beta t)$$

where

$$\alpha = \frac{\omega + \sqrt{3}}{2} \quad \text{and} \quad \beta = \frac{\omega - \sqrt{3}}{2}.$$

Here is the graph of this solution in the case where $\omega = 1.6$. For this value of ω, $\alpha = 1.67$ and $\beta = -0.066$.



What happens if $\omega = \sqrt{3}\,$?

Example.

$$\frac{d^2y}{dt^2} + 3y = \cos\sqrt{3}\,t$$

The complexified equation is

$$\frac{d^2y}{dt^2} + 3y = e^{i\sqrt{3}\,t}.$$

What guess should we use?

(Additional blank space on the top of the next page.)

Here is the graph of $y_p(t)$.