Conserved quantities

When we discussed linearization, we saw that there was one situation where the linearized system could not predict the behavior of the nonlinear system near an equilibrium point. If the eigenvalues of the Jacobian matrix are purely imaginary, then the Linearization Theorem does not apply.

Consider the predator-prey system

$$\begin{aligned}\frac{dR}{dt} &= R - 0.5RF \\ \frac{dF}{dt} &= -F + 0.25RF.\end{aligned}$$

This system has an equilibrium point at $(R, F) = (4, 2)$.

How often do systems have conserved quantities?

Hamilton observed that many systems from classical mechanics have some type of conserved quantity. His observation led to the following definition:

**Definition.** A system of differential equations

$$\frac{dx}{dt} = P(x, y)$$
$$\frac{dy}{dt} = Q(x, y)$$

is a Hamiltonian system if there exists a function $H(x, y)$ such that

$$P(x, y) = \frac{\partial H}{\partial y} \qquad \text{and} \qquad Q(x, y) = -\frac{\partial H}{\partial x}.$$

We say that $H(x, y)$ is the Hamiltonian function for the system.

**Theorem.** If a system is a Hamiltonian system with Hamiltonian function $H(x, y)$, then $H(x, y)$ is a conserved quantity for the system.

**Example.** Consider the ideal pendulum

$$\frac{d\theta}{dt} = v$$

$$\frac{dv}{dt} = -k \sin \theta.$$

This system is a Hamiltonian system with Hamiltonian

$$H(\theta, v) = \tfrac{1}{2}v^2 - k \cos \theta.$$

Is there a way to tell if a system

$$\frac{dx}{dt} = P(x,y)$$
$$\frac{dy}{dt} = Q(x,y)$$

is a Hamiltonian system?

Necessary condition:

$$\frac{\partial P}{\partial x} = -\frac{\partial Q}{\partial y}$$

**Example.** Consider the linear system

$$\frac{dx}{dt} = x + 4y$$
$$\frac{dy}{dt} = -2x - y.$$

Extreme values of the Hamiltonian

**Theorem.** Let

$$\frac{d\mathbf{Y}}{dt} = \mathbf{F}(\mathbf{Y})$$

be a Hamiltonian system with Hamiltonian function $H$. If $\mathbf{Y}_0$ is an isolated equilibrium point that is a local extreme value for $H$, then all solution curves near $\mathbf{Y}_0$ are closed curves.

Dissipative systems

Consider the damped pendulum

$$\frac{d\theta}{dt} = v$$

$$\frac{dv}{dt} = -\sin\theta - kv.$$

This system is not a Hamiltonian system. However, if the damping term were not present, then the system would be Hamiltonian with

$$H(\theta, v) = \tfrac{1}{2}v^2 - \cos\theta.$$

**Definition.** A function $L(x, y)$ is a Lyapunov function for a system of differential equations if

$$\frac{d}{dt} L(x(t), y(t)) \leq 0$$

for all solutions $(x(t), y(t))$ that are not equilibrium solutions and for all $t$ with strict inequality except for a discrete set of values of $t$.

In general it is not easy to determine if a system has a Lyapunov function. However, there is a class of systems that come with a Lyapunov function that is immediately evident.

**Definition.** A system of differential equations is a gradient system if there is a function $G(x, y)$ such that

$$\frac{dx}{dt} = \frac{\partial G}{\partial x}$$
$$\frac{dy}{dt} = \frac{\partial G}{\partial y}$$

for all $(x, y)$.

**Example.** Let $G(x, y) = 2x^2 - x^4 - 2y^2$. The gradient system of $G(x, y)$ is

$$\frac{dx}{dt} = 4x - 4x^3$$
$$\frac{dy}{dt} = -4y.$$