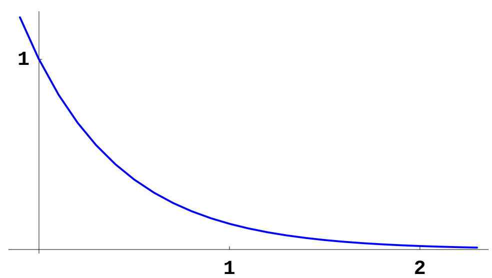
The Laplace transform

For the remainder of the semester, we are going to take a somewhat different approach to the solution of differential equations. We are going to study a way of transforming differential equations into algebraic equations.

We begin with a little review of improper integrals.

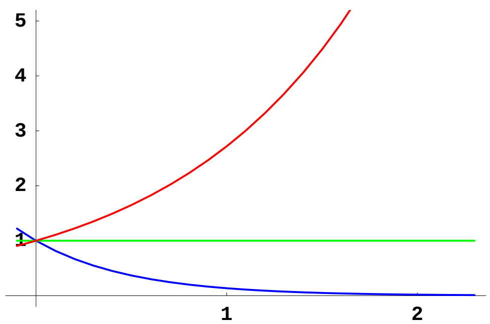
Example. Consider the improper integral

$$\int_0^\infty e^{-2t}\,dt.$$



Example. Consider the improper integrals

$$\int_0^\infty e^{-st}\,dt.$$



Definition. The *Laplace transform* of the function $y(t)$ is the function

$$Y(s) = \int_0^\infty y(t)\,e^{-st}\,dt.$$

This transform is an "operator" (a function on functions). It transforms the function $y(t)$ into the function $Y(s)$.

Notation: We often represent this operator using the script letter $\mathcal{L}$. In other words,

$$\mathcal{L}[y] = Y.$$

For example, $\mathcal{L}[1] = \dfrac{1}{s}$.

Note that, even if $y(t)$ is defined for all t, the Laplace transform $Y(s)$ may not be defined for all s. In fact, there are functions such as $y(t) = e^{t^2}$ for which the improper integral does not exist for any s.

Theorem. Suppose that the function $y(t)$ has only a finite number of jump discontinuities in any finite interval. In addition, suppose that $y(t)$ grows no faster than a given exponential function. In other words, suppose that there exist constants K and M such that

$$|y(t)| \leq e^{Mt} \quad \text{for} \quad t \geq K.$$

Then $\mathcal{L}[y]$ exists for $s > M$.

Example. Let's compute $\mathcal{L}[e^{at}]$ using the definition and the improper integrals we have already computed:

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{at} e^{-st}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \frac{1}{s-a} \quad \text{for} \quad s > a.$$

Examples. Using *Mathematica* to calculate the improper integrals, we see that:

$$\mathcal{L}[\sin t] = \frac{1}{s^2+1} \quad \text{for} \quad s > 0$$

$$\mathcal{L}[e^{2t} \sin 3t] = \frac{3}{s^2 - 4s + 13} \quad \text{for} \quad s > 2$$

$$\mathcal{L}[t^4] = \frac{24}{s^5} \quad \text{for} \quad s > 0$$

$$\mathcal{L}[\sin 2t] = \frac{2}{s^2+4} \quad \text{for} \quad s > 0,$$

$$\mathcal{L}[t \cos \sqrt{2}\, t] = \frac{s^2-2}{(s^2+2)^2} \quad \text{for} \quad s > 0$$

$$\mathcal{L}[e^{i\omega t}] = \frac{1}{s - i\omega} \quad \text{for} \quad s > 0$$

Properties of the Laplace transform There are two properties of the Laplace transform that make it well suited for solving linear differential equations:

1. $\mathcal{L}\left[\frac{dy}{dt}\right] = s\mathcal{L}[y] - y(0)$

2. $\mathcal{L}$ is a linear transform

Both of these properties are extremely important, but the surprising one is #1. Let's consider

$$\mathcal{L}\left[\frac{dy}{dt}\right] = \int_0^\infty \left(\frac{dy}{dt}\right) e^{-st}\,dt.$$

In fact, before we consider the improper integral, let's apply the method of integration by parts to the indefinite integral

$$\int \left(\frac{dy}{dt}\right) e^{-st}\,dt.$$

Now let's see how we can use the Laplace transform to solve an initial-value problem.

Example. Solve the initial-value problem

$$\frac{dy}{dt} - 3y = e^{2t}, \quad y(0) = 4.$$

1. Transform both sides of the equation:

2. Solve for $\mathcal{L}[y]$:

3. Calculate the inverse Laplace transform:

Is this the right answer? Do we need Laplace transforms to calculate it?

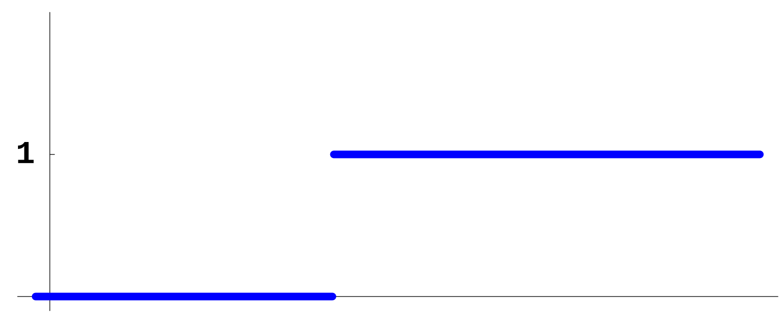
Discontinuous differential equations

The Laplace transform works well on linear differential equations that are discontinuous in one way or another.

Definition. The *Heaviside function* $u_a(t)$ is the function defined by

$$u_a(t) = \begin{cases} 0, & \text{if } t < a; \\ 1, & \text{if } t \geq a. \end{cases}$$

Thus $u_a(t)$ has a discontinuity at $t = a$ where it jumps from 0 to 1. Note that the `step(t)` function in `DETools` is the same function as $u_0(t)$.



Here's how you can use the Heaviside function to avoid piecewise definitions:

Example. Consider $g(t) = 2t + u_1(t)(2 - 2t)$.

Laplace transforms are very convenient if we have discontinuous forcing. How do we calculate the Laplace transform of a discontinuous function?

Example. Let's calculate $\mathcal{L}[u_a]$ directly from the definition of $\mathcal{L}$.

In order to calculate inverse Laplace transforms, we need another property of the transform.

Rule 3: Shifting the t-axis. $\mathcal{L}[u_a(t) f(t-a)] = e^{-as}\mathcal{L}[f]$.

Example. Calculate $\mathcal{L}[g]$ where $g(t) = u_2(t)\, e^{-(t-2)}$.