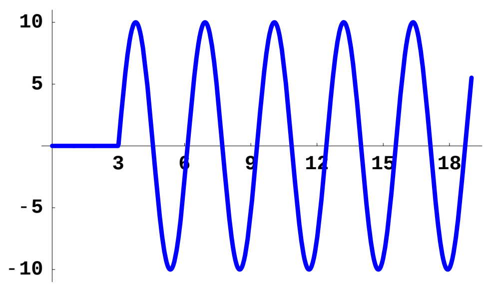
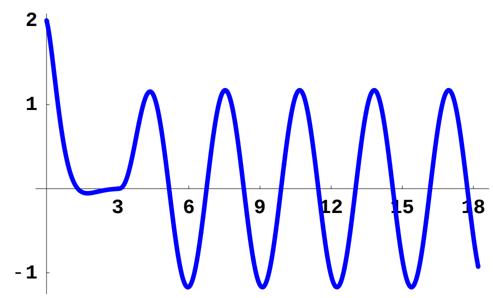
More on using the Laplace transform to solve certain second-order equations

Let's solve the initial-value problem

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 7y = 10\,u_3(t)\,\sin 2(t-3), \quad y(0) = 2, \quad y'(0) = -1.$$

Before we get too far into the messy formulas, let's look at the graphs of the forcing function $10u_3(t)\sin 2(t-3)$ and the solution:





Now for the formulas:

1. Transform both sides of the equation:

2. Solve for $\mathcal{L}[y]$:

3. Calculate the inverse Laplace transform:

We calculated

$$\mathcal{L}^{-1}\left[\frac{2s+7}{s^2+4s+7}\right] = 2e^{-2t}\cos\sqrt{3}\,t + \sqrt{3}\,e^{-2t}\sin\sqrt{3}\,t$$

in Example 2.

To invert the second term, we take advantage of some algebra done before class:

(a) Partial fractions decomposition:

$$\frac{1}{(s^2+4)(s^2+4s+7)} = \frac{1}{73}\left(\frac{4s+13}{s^2+4s+7} - \frac{4s-3}{s^2+4}\right)$$

(b) Inverse related to the first term:

$$\mathcal{L}^{-1}\left[\frac{4s+13}{s^2+4s+7}\right] = 4e^{-2t}\cos\sqrt{3}\,t + \frac{5\sqrt{3}}{3}e^{-2t}\sin\sqrt{3}\,t$$

(c) Inverse related to the second term:

$$\mathcal{L}^{-1}\left[\frac{4s-3}{s^2+4}\right] = 4\cos 2t - \frac{3}{2}\sin 2t$$

After we put all of this together, we get the solution

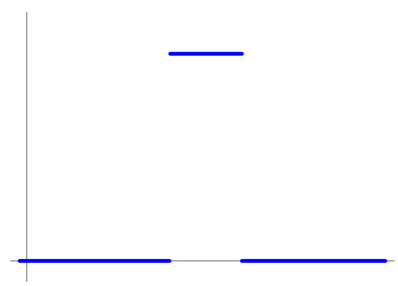
$$y(t) = 2e^{-2t}\cos\sqrt{3}\,t + \sqrt{3}\,e^{-2t}\sin\sqrt{3}\,t +$$

$$\frac{20}{73}\,u_3(t)\left(4e^{-2(t-3)}\cos\sqrt{3}(t-3) + \frac{5\sqrt{3}}{3}e^{-2(t-3)}\sin\sqrt{3}(t-3) - 4\cos 2(t-3) + \frac{3}{2}\sin 2(t-3)\right)$$

Dirac Delta Function

The Dirac delta "function" $\delta_a(t)$ is used to model impulse forcing. In other words, suppose we want to model a unit force that is applied instantaneously at time $t = a$. We begin with the function

$$g_{\Delta t}(t) = \begin{cases} \dfrac{1}{\Delta t}, & \text{if } a - \dfrac{\Delta t}{2} \leq t < a + \dfrac{\Delta t}{2}; \\ 0, & \text{otherwise.} \end{cases}$$



We can write $g_{\Delta t}$ in terms of the Heaviside function. We get

$$g_{\Delta t}(t) = \frac{1}{\Delta t}\left(u_{a-\frac{\Delta t}{2}}(t) - u_{a+\frac{\Delta t}{2}}(t)\right).$$

Let's calculate the Laplace transform of $g_{\Delta t}$. To do so, we'll need the limit

$$\lim_{x \to 0} \frac{e^{sx} - 1}{x} =$$

This limit can be calculated using L'Hospital's Rule, using power series, or by observing that this limit is simply $f'(0)$ for the function $f(x) = e^{sx}$.

Now we calculate the Laplace transform of $g_{\Delta t}$:

We take the limit as $\Delta t \to 0$.

Dirac Delta Function. The Dirac delta function $\delta_a(t)$ is the "function" such that

$$\mathcal{L}[\delta_a] = e^{-as}.$$

Example. Consider the initial-value problem

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 7y = 15\,\delta_5(t), \quad y(0) = 2, \quad y'(0) = -1.$$

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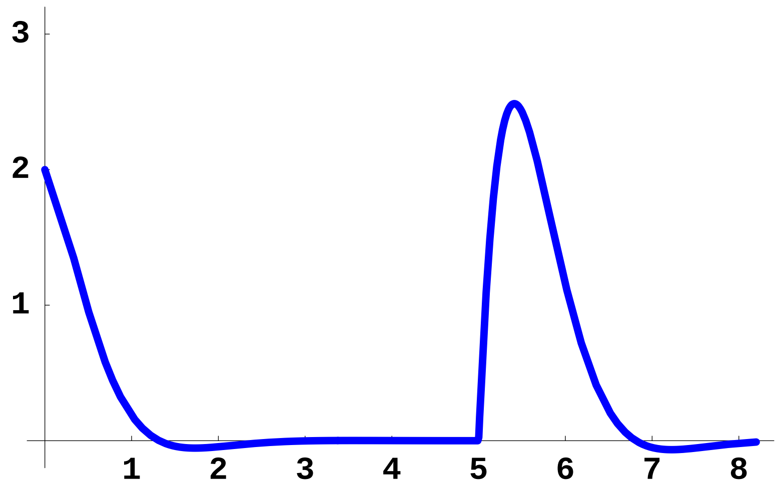
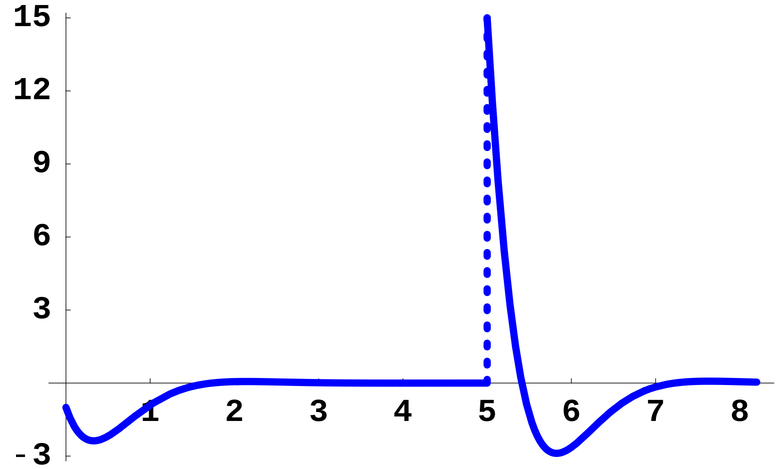
Here is the graph of the solution:



Here is the graph of its derivative:



Summary of transform rules and table of standard transforms

Here are important selections from the summary on page 620 in your text.

$y(t)$	$Y(s) = \mathcal{L}[y]$	
$y(t) = 1$	$Y(s) = \dfrac{1}{s}$	$(s > 0)$
$y(t) = e^{at}$	$Y(s) = \dfrac{1}{s-a}$	$(s > a)$
$y(t) = u_a(t)$	$Y(s) = \dfrac{e^{-as}}{s}$	$(s > 0)$
$y(t) = \cos\omega t$	$Y(s) = \dfrac{s}{s^2+\omega^2}$	$(s > 0)$
$y(t) = \sin\omega t$	$Y(s) = \dfrac{\omega}{s^2+\omega^2}$	$(s > 0)$
$y(t) = \delta_a(t)$	$Y(s) = e^{-as}$	

Properties of the Laplace Transform

$$\mathcal{L}\left[\frac{dy}{dt}\right] = s\mathcal{L}[y] - y(0)$$

$$\mathcal{L}[y_1 + y_2] = \mathcal{L}[y_1] + \mathcal{L}[y_2]$$

$$\mathcal{L}[\alpha\, y] = \alpha\mathcal{L}[y] \text{ for any constant } \alpha$$

$$\mathcal{L}[u_a(t)y(t-a)] = e^{-as}\mathcal{L}[y]$$

$$\mathcal{L}[e^{at}y(t)] = Y(s-a) \text{ where } Y = \mathcal{L}[y]$$

Some people like to memorize a few more entries such as

$$\mathcal{L}[e^{at}\cos\omega t] = \frac{s-a}{(s-a)^2+\omega^2},$$

but I prefer to use the last rule (shifting the s-axis). Also, the rule for $\mathcal{L}[\frac{dy}{dt}]$ in terms of $\mathcal{L}[y]$ yields

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] = s^2\mathcal{L}[y] - y(0)s - y'(0).$$

Warning: Just because you can solve a linear differential equation with the Laplace transform does not mean that you should forget what you learned in previous parts of the course. The transform method is particularly well suited for differential equations with discontinuous and impulse forcing.