More on the error in Euler's method

Last class we analyzed the error e_1 involved in the first step of Euler's method, and we obtained the estimate

$$e_1 \leq (M_1)\frac{(\Delta t)^2}{2},$$

where M_1 is a bound on the quantity $\left|\frac{\partial f}{\partial t} + \left(\frac{\partial f}{\partial y}\right) f(t,y)\right|$.

The error in the second step:

Note that the analysis of the error in this step is complicated by the fact that the point (t_1, y_1) is not necessarily on the graph of the solution $y(t)$.

The error in the kth step ($k \geq 2$):

In general we obtain a recursive formula for the error e_k in terms of the error e_{k-1}. We have

$$e_k \leq (1 + M_2 \Delta t) e_{k-1} + M_1 \frac{(\Delta t)^2}{2}.$$

In Exercise 11 of Section 7.1, we show that this recursive formula yields the following theorem.

Theorem. Given the bounds M_1 and M_2 as above, then the error

$$e_n \leq (C)(\Delta t),$$

where C is a constant that is determined by M_1, M_2, and the length of the interval over which the solution is approximated.

Note that the constant C does not depend on the number of steps used. Another way to write this result is as

$$e_n \leq (C)(\Delta t) = (C)\left(\frac{t_n - t_0}{n}\right) = \frac{K}{n}.$$

Euler's method is the most basic "fixed-step-size" algorithm for numerically approximating solutions. `HPGSolver` also uses a fixed-step-size algorithm called the Runge-Kutta method. The Runge-Kutta method is usually more efficient and more accurate than Euler's method (see Section 7.3 of our text). Unfortunately, there are differential equations that are not amenable to fixed-step-size algorithms.

Example. Consider the initial-value problem

$$\frac{dy}{dt} = e^t \sin y, \quad y(0) = 5.$$

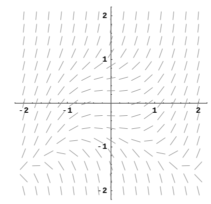
Let's see what happens when we use Euler's method to approximate the solution with various step sizes $0.01 \leq \Delta t \leq 0.1$.

The spreadsheet for this example is also posted on the course web site.

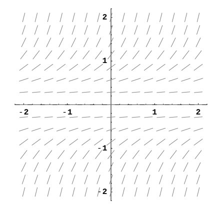
Existence and Uniqueness Theory

First we consider three examples to illustrate the idea of the domain of a differential equation:

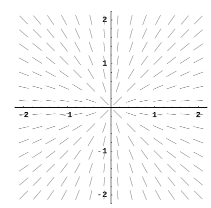
Example 1. $\dfrac{dy}{dt} = y^3 + t^2$



Example 2. $\dfrac{dy}{dt} = y^2$



Example 3. $\dfrac{dy}{dt} = \dfrac{y}{t}$



We start our discussion of the theory with the Existence Theorem:

Existence Theorem Suppose $f(t, y)$ is a continuous function in a rectangle of the form

$$\{(t, y) \mid a < t < b,\ c < y < d\}$$

in the ty-plane. If (t_0, y_0) is a point in this rectangle, then there exists an $\epsilon > 0$ and a function $y(t)$ defined for

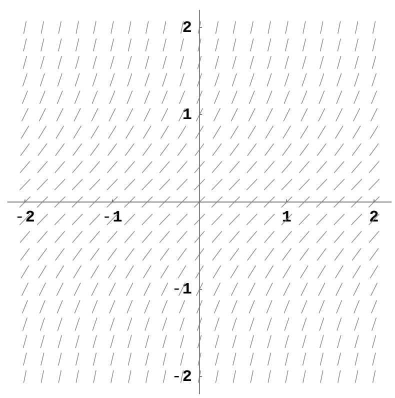
$$t_0 - \epsilon < t < t_0 + \epsilon$$

that solves the initial-value problem

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0. \quad \blacksquare$$

What's the significance of the ϵ in the Existence Theorem?

Example. $\frac{dy}{dt} = 1 + y^2, \quad y(0) = 0$



What does the Existence Theorem tell us about the initial-value problem

$$\frac{dy}{dt} = y^3 + t^2, \quad y(0) = 0?$$

Picard iteration

How do we know that solutions exist if we cannot express them in closed form?

Consider the initial-value problem

$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0.$$

We can use the Fundamental Theorem of Calculus to rewrite this differential equation as an integral equation.

Picard iteration: Modify this integral equation so that it produces an iterative procedure. Start with the constant function

$$y_0(t) = y_0 \quad \text{for all } t.$$

Then produce an infinite sequence of functions

$$y_{k+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_k(s))\, ds.$$

Does this infinite sequence converge to a limiting function?

Example. Let's apply this iterative procedure to the initial-value problem

$$\frac{dy}{dt} = y, \quad y(0) = 1.$$

What happens if the Picard iterates converge to a function $y_*(t)$? In other words, suppose that

$$\lim_{k\to\infty} y_k(t) = y_*(t).$$