Picard iteration

How do we know that solutions exist if we cannot express them in closed form?

Consider the initial-value problem

$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0.$$

We can use the Fundamental Theorem of Calculus to rewrite this differential equation as an integral equation.

Picard iteration: Modify this integral equation so that it produces an iterative procedure. Start with the constant function

$$y_0(t) = y_0 \quad \text{for all } t.$$

Then produce an infinite sequence of functions

$$y_{k+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_k(s))\, ds.$$

Does this infinite sequence converge to a limiting function?

Example. Let's apply this iterative procedure to the initial-value problem

$$\frac{dy}{dt} = y, \quad y(0) = 1.$$

What happens if the Picard iterates converge to a function $y_*(t)$? In other words, suppose that

$$\lim_{k\to\infty} y_k(t) = y_*(t).$$

Uniqueness

Uniqueness Theorem Suppose $f(t, y)$ and $\partial f/\partial y$ are continuous functions in a rectangle of the form

$$\{(t, y) \mid a < t < b,\ c < y < d\}$$

in the ty-plane. If (t_0, y_0) is a point in this rectangle and if $y_1(t)$ and $y_2(t)$ are two functions that solve the initial-value problem

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

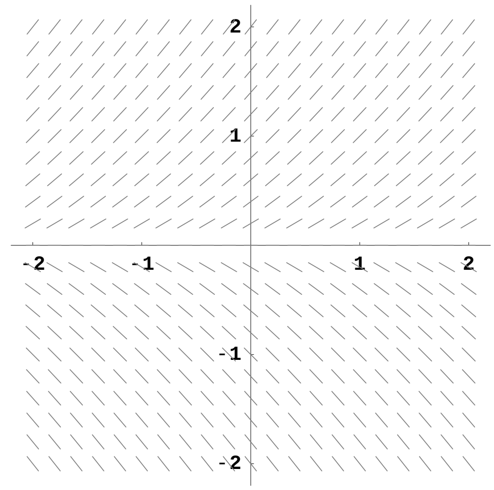
for all t in the interval $t_0 - \epsilon < t < t_0 + \epsilon$ (where ϵ is some positive number), then

$$y_1(t) = y_2(t)$$

for $t_0 - \epsilon < t < t_0 + \epsilon$. That is, the solution to the initial-value problem is *unique*. ∎

Here's an example that lacks uniqueness:

Example. $\dfrac{dy}{dt} = \sqrt[3]{y}$



Bogus Example. The example

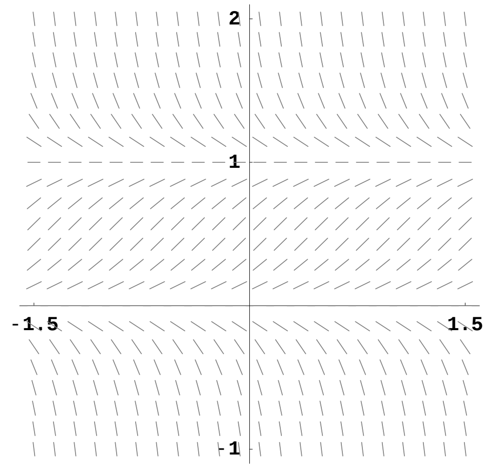
$$\frac{dy}{dt} = \frac{y}{t} + t\cos t$$

in `FirstOrderSystems` seems to violate the Uniqueness Theorem, but in fact it does not. Why?

The Uniqueness Theorem has many useful consequences. Here are three examples:

Example 1. $\dfrac{dy}{dt} = -2ty^2$

Example 2. $\dfrac{dy}{dt} = 4y(1-y)$



Example 3. $\dfrac{dy}{dt} = e^t \sin y$