Linear differential equations

A first-order differential equation

$$\frac{dy}{dt} = f(t, y)$$

with independent variable t and dependent variable y is **linear** if it can be written as

$$\frac{dy}{dt} = a(t)y + b(t).$$

In other words, the *dependent* variable only appears linearly in the equation.

Linear differential equations:

$$\frac{dy}{dt} = 5y$$

$$\frac{dy}{dt} = (\cos t)y$$

$$\frac{dy}{dt} = y - t^2$$

Nonlinear differential equations:

$$\frac{dy}{dt} = t \cos y$$

$$\frac{dy}{dt} = y^2 - t$$

The linear differential equation

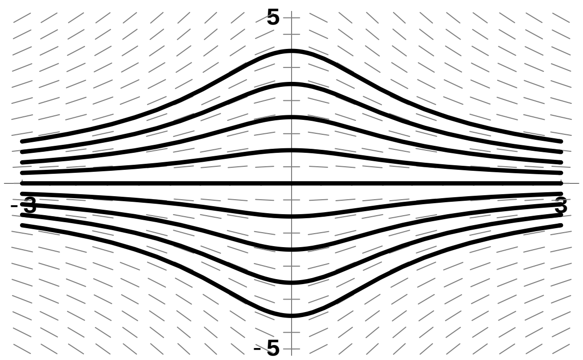
$$\frac{dy}{dt} = a(t)y + b(t)$$

is **homogeneous** if $b(t) = 0$ for all t. Otherwise, it is **nonhomogeneous**. (Some people use the term inhomogeneous.)

Where have we seen homogeneous linear differential equations before?

Example. $\dfrac{dy}{dt} = \dfrac{-ty}{1+t^2}$

(Graphs and slope field on top of next page.)



Linearity Principles

Why are linear equations so much more amenable to analytic techniques than nonlinear equations? The reason is that their solutions satisfy important linearity principles.

Let's begin with homogeneous linear equations:

Linearity Principle. If $y_h(t)$ is a solution of a homogeneous linear differential equation

$$\frac{dy}{dt} = a(t)y,$$

then any *constant* multiple $y_k(t) = ky_h(t)$ of $y_h(t)$ is also a solution. In other words, given a constant $k \neq 1$ and a solution $y_h(t)$, we obtain another solution by multiplying $y_h(t)$ by k.

Note that the Linearity Principle is not true for nonlinear equations. For example, consider

$$\frac{dy}{dt} = y^2.$$

Check that one solution is

$$y_1(t) = \frac{1}{1-t},$$

and then check that

$$y_2(t) = 2y_1(t) = \frac{2}{1-t}$$

is not a solution.

There is a similar "linearity" principle for nonhomogeneous linear equations:

Extended Linearity Principle For First-Order Equations. Consider a first-order, nonhomogeneous, linear equation

$$\frac{dy}{dt} = a(t)y + b(t)$$

and its associated homogeneous equation

$$\frac{dy}{dt} = a(t)y.$$

1. If $y_h(t)$ is any solution of the homogeneous equation and $y_p(t)$ ("p" for particular solution) is *any* solution of the nonhomogeneous equation, then $y_h(t) + y_p(t)$ is also a solution of the nonhomogeneous equation.

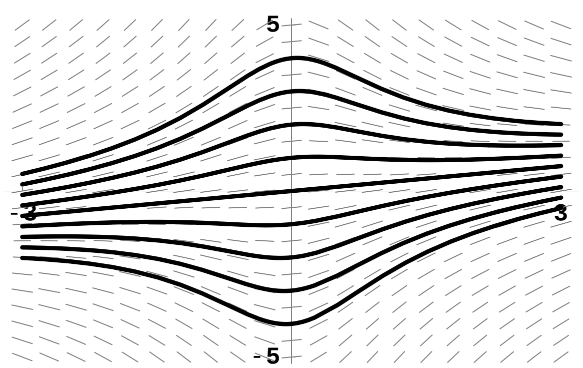
2. Suppose $y_p(t)$ and $y_q(t)$ are two solutions of the nonhomogeneous equation. Then $y_p(t) - y_q(t)$ is a solution of the associated homogeneous equation.

Therefore, if $y_h(t)$ is nonzero, $ky_h(t) + y_p(t)$ is the general solution of the nonhomogeneous equation.

We can paraphrase the Extended Linearity Principle by saying that:

> The general solution of a nonhomogeneous linear equation consists of the sum of *any* particular solution of the nonhomogeneous equation and the general solution of the associated homogeneous equation.

Example. $\dfrac{dy}{dt} = \dfrac{-ty}{1+t^2} + \dfrac{2t^2+1}{4t^2+4}$



The Extended Linearity Principle provides one way to solve nonhomogeneous equations:

Method of the Lucky Guess:

1. Solve the associated homogeneous equation.
2. Guess one solution to the nonhomogeneous equation.

How do we go about guessing a solution? This question is best answered by doing a number of typical examples.

Example 1. $\dfrac{dy}{dt} = -2y + 3e^{-t/2}$

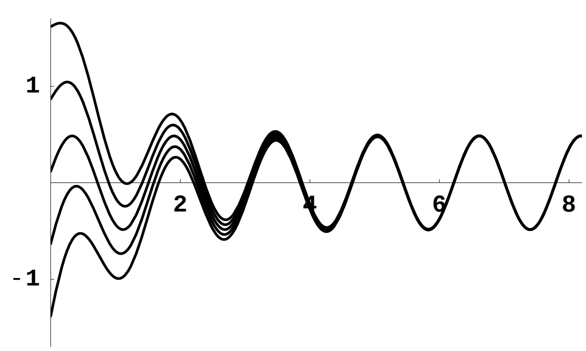
1. General solution of the associated homogeneous equation:

2. Particular solution of the nonhomogeneous equation:

Example 2. $\dfrac{dy}{dt} = -y + 2\cos 4t$

1. General solution of the associated homogeneous equation:

2. Particular solution of the nonhomogeneous equation:



Example 3. $\dfrac{dy}{dt} = -3y + 2e^{-3t}$

1. General solution of the associated homogeneous equation:

2. Particular solution of the nonhomogeneous equation (trick question):