1. (18 points) Let

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}.$$

(a) Show that $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal.

$$u_1 \cdot u_2 = (1)(4) + (-2)(1) + (1)(-2) = 0$$

(b) Compute the orthogonal projection of $\mathbf{e}_3$ onto the span of $\{\mathbf{u}_1, \mathbf{u}_2\}$. Why does this computation show that $\mathbf{e}_3$ is not in this plane?

$W = \text{span}\{u_1, u_2\}$

$$\text{proj}_W e_3 = \left(\frac{e_3 \cdot u_1}{u_1 \cdot u_1}\right) u_1 + \left(\frac{e_3 \cdot u_2}{u_2 \cdot u_2}\right) u_2$$

$$= \frac{1}{6} u_1 + \frac{-2}{21} u_2$$

$$= \begin{bmatrix} \frac{1}{6} \\ -\frac{1}{3} \\ \frac{1}{6} \end{bmatrix} + \begin{bmatrix} -\frac{8}{21} \\ -\frac{2}{21} \\ \frac{4}{21} \end{bmatrix} = \begin{bmatrix} -\frac{3}{14} \\ -\frac{3}{7} \\ \frac{5}{14} \end{bmatrix}$$

$e_3 \notin W$ because

$\text{proj}_W e_3 \neq e_3$.

(c) Find a vector $\mathbf{u}_3$ in $\mathbb{R}^3$ such that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis of $\mathbb{R}^3$.

$$u_3 = e_3 - \text{proj}_W e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -\frac{3}{14} \\ -\frac{3}{7} \\ \frac{5}{14} \end{bmatrix} = \begin{bmatrix} \frac{3}{14} \\ \frac{3}{7} \\ \frac{9}{14} \end{bmatrix}$$

2. (16 points) Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ -2 & 6 & -6 & -1 & -10 \\ -3 & 9 & -6 & -6 & -3 \\ 3 & -9 & 4 & 9 & 0 \end{bmatrix}.$$

Find a basis of the orthogonal complement of Nul **A**. What is the dimension of this complement?

$(\text{Nul} A)^{\perp} = \text{Row} A$

Want a basis of Row A:

$$A \sim \begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ 0 & 0 & 2 & -3 & 8 \\ 0 & 0 & 6 & -9 & 24 \\ 0 & 0 & -8 & 12 & -27 \end{bmatrix} \quad \text{by } \begin{array}{l} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 + 3R_1 \\ R_4 \to R_4 - 3R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ 0 & 0 & 2 & -3 & 8 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} \quad \text{by } \begin{array}{l} R_3 \to R_3 - 3R_2 \\ R_4 \to R_4 + 4R_2 \end{array}$$

Then flip rows 3 and 4 to get REF.

basis of Row A $= \left\{ \begin{bmatrix} 1 \\ -3 \\ 4 \\ -1 \\ 9 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ -3 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 5 \end{bmatrix} \right\}$

$\Rightarrow \dim(\text{Row} A) = 3$

3. (12 points) Suppose a consistent nonhomogeneous system $\mathbf{Ax} = \mathbf{b}_1$ of seven linear equations in nine unknowns has a solution set that contains two free variables. Is it possible to find a $\mathbf{b}_2$ such that the system $\mathbf{Ax} = \mathbf{b}_2$ is inconsistent? (In order to receive any credit, you must provide a valid justification for your answer.)

seven equations in nine unknowns

$\Rightarrow A$ is 7×9 matrix.

Because $Ax = b_1$ has a solution set with two free variables, $\dim \operatorname{Nul} A = 2$.

$\Rightarrow \dim \operatorname{Row} A = 9 - 2 = 7$.

$\Rightarrow \operatorname{rank} A = 7$

$\Rightarrow \dim \operatorname{Col} A = 7$.

Therefore, $\operatorname{Col} A$ is a seven dimensional subspace of $\mathbb{R}^7$. The only such subspace is $\mathbb{R}^7$ itself.

$\Rightarrow \operatorname{Col} A = \mathbb{R}^7$

$\Rightarrow$ the system $Ax = b$ is consistent for any $b \in \mathbb{R}^7$.

Final answer: No, it is not possible to find a b_2 such that $Ax = b_2$ is inconsistent.

4. (16 points) Consider

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ -1 & 5 \end{bmatrix}, \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

(a) Show that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors for the matrix $\mathbf{A}$.

$$Av_1 = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad Av_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

eigenvalues are 2 and 4

(b) Compute $\mathbf{A}^k$ for all positive integers k.

$$A = PDP^{-1} = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}\left(-\tfrac{1}{2}\right)\begin{bmatrix} 1 & -3 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\Rightarrow A^k = PD^kP^{-1} = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4^k & 0 \\ 0 & 2^k \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 4^k & 3(2^k) \\ 4^k & 2^k \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{3}{2}\right)(2^k) - \left(\frac{1}{2}\right)4^k & \left(\frac{3}{2}\right)4^k - \left(\frac{3}{2}\right)2^k \\ \left(\frac{1}{2}\right)2^k - \left(\frac{1}{2}\right)4^k & \left(\frac{3}{2}\right)4^k - \left(\frac{1}{2}\right)2^k \end{bmatrix}$$

$$= \begin{bmatrix} 3(2^{k-1}) - \cancel{4}\,2^{2k-1} & 3(2^{2k-1}) - 3(2^{k-1}) \\ 2^{k-1} - 2^{2k-1} & 3(2^{2k-1}) - 2^{k-1} \end{bmatrix}$$

5. (18 points) Which of the following three matrices can be diagonalized? In order to receive credit, you must justify your answer, but **you do not have to diagonalize the matrices that can be diagonalized.**

(a) $\mathbf{A} = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$ char poly $\det \begin{bmatrix} 3-\lambda & -1 \\ 1 & 3-\lambda \end{bmatrix} = (\lambda-3)^2 + 1$

The char poly is at least 1 for all real λ, and therefore it does not have any real roots (no real eigenvalues).
$\Rightarrow$ A is not diagonalizable.

(b) $\mathbf{B} = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}$

char poly $\det \begin{bmatrix} -\lambda & 1 \\ -4 & -4-\lambda \end{bmatrix} = (\lambda+4)\lambda + 4$
$= \lambda^2 + 4\lambda + 4$
$= (\lambda+2)^2 \Rightarrow$
$\lambda = -2$ repeated eval.

Need $\dim \mathrm{Nul}(B+2I)$:

$B+2I = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$

$\mathrm{rank}(B+2I) = 1 \Rightarrow$
$\dim(\mathrm{Nul}(B+2I)) = 1 \Rightarrow$
no basis of $\mathbb{R}^2$ of evecs $\Rightarrow$ not diag

(c) $\mathbf{C} = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$

char poly $\det \begin{bmatrix} 4-\lambda & 2 \\ 2 & 2-\lambda \end{bmatrix} = (\lambda-4)(\lambda-2) - 4$
$= \lambda^2 - 6\lambda + 4$

eigenvalues: $\lambda = \dfrac{3 \pm \sqrt{36-16}}{2}$

two distinct, real eigenvalues for a 2x2 matrix $\Rightarrow$ C is diagonalizable.

6. (20 points) Are the following statements true or false? **You will not receive any credit unless you justify your answers.** (Note that there are two more parts to this question on the next page.)

(a) Every orthonormal set is orthogonal.

True. An orthonormal set $\{v_1, v_2, \dots, v_k\}$ satisfies the conditions $v_i \cdot v_j = 0$ for $i \neq j$ and $v_i \cdot v_i = 1$ for $i = 1, 2, \dots, k$. An orthogonal set satisfies $v_i \cdot v_j = 0$ for $i \neq j$. So every orthonormal set is orthogonal, but not every orthogonal set is orthonormal.

(b) If a matrix $\mathbf{U}$ has orthogonal columns, then $\mathbf{U}^T\mathbf{U} = \mathbf{I}$.

False. The matrix $U = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

has orthogonal columns, but

$$U^T U = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Question 6 (continued):

(c) If $\mathbf{A}$ and $\mathbf{B}$ are row equivalent square matrices, then the eigenvalues of $\mathbf{A}$ are the same as the eigenvalues of $\mathbf{B}$.

False. The matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ are row equivalent (both A and B are row equivalent to I), but the eigenvalues of A are 1 and 2 and the eigenvalues of B are 3 and 4.

(d) If $\mathbf{A}$ and $\mathbf{B}$ are row equivalent matrices, then $\operatorname{rank} \mathbf{A} = \operatorname{rank} \mathbf{B}$.

True. Row operations on a matrix do not change the row space. That is, if $A \sim B$, then $\operatorname{row} A = \operatorname{row} B$. Since the rank of a matrix is equal to the dimension of its row space, $\operatorname{rank} A = \operatorname{rank} B$ if $A \sim B$.