Subspaces of vector spaces

**Definition.** A nonempty subset \( S \) of a vector space \( V \) is a *subspace* of \( V \) if

1. the zero vector \( 0 \) is in \( S \),

2. (closure under vector addition) for each \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) in \( S \), the vector sum \( \mathbf{v}_1 + \mathbf{v}_2 \) is in \( S \), and

3. (closure under scalar multiplication) for each \( r \) in \( \mathbb{R} \) and each \( \mathbf{v} \) in \( S \), the scalar multiple \( r\mathbf{v} \) is in \( S \).

**Note.** A subspace \( S \) of a vector space \( V \) is a vector space in its own right.

**Example.** Consider the line \( x_2 = 3x_1 \) in the vector space \( \mathbb{R}^2 \).

**Example.** Consider the line \( x_2 = x_1 + 1 \) in the vector space \( \mathbb{R}^2 \).
Example. Let $\mathbb{P}$ represent the vector space of all polynomial functions as discussed last class. Is $\mathbb{P}$ a subspace of the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$?

Example. Consider the subset $S = \text{Span}\{x, x^2\}$ within $\mathbb{P}$. Is $S$ a subspace of $\mathbb{P}$?

Theorem. If $v_1, v_2, \ldots, v_p$ are vectors in a vector space $V$, then $\text{Span}\{v_1, v_2, \ldots, v_p\}$ is a subspace of $V$. 
Example. Let $V$ be the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$. Which of the following subsets of $V$ are subspaces of $V$?

1. The set of all constant functions.

2. The set of all functions $f$ such that $f(2) = 1$.

3. The set of all functions $f$ such that $f(2) = 0$.

4. The set of all polynomials of degree 3.

5. The set of all polynomials whose degree is at most 3.

6. The set of all differentiable functions.
Subspaces associated to a matrix

There are three important subspaces associated to an $m \times n$ matrix $A$.

**The null space of $A$.** The null space of $A$ is the set of all vectors $x$ in $\mathbb{R}^n$ such that

$$Ax = 0.$$ 

The null space of $A$ is denoted by $\text{Nul } A$.

**Theorem.** The null space of an $m \times n$ matrix $A$ is a subspace of $\mathbb{R}^n$. 

Application. Any plane through the origin in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.

Example. Consider the matrix

$$A = \begin{bmatrix} 1 & -2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -9 & 0 \\ 2 & -4 & 0 & 8 & 1 \end{bmatrix}.$$  

Express the null space of $A$ as the span of as few vectors as possible.