Matrix multiplication

Recall our definition of the product $AB$ of two matrices.

**Definition.** If $A$ is an $m \times n$ matrix and $B$ is an $n \times p$ matrix, then the product $AB$ is the matrix

$$AB = \begin{bmatrix} AB_1 & AB_2 & \ldots & AB_p \end{bmatrix},$$

where $B_j$ represents the $j$th column of $B$.

**Row-column dot product definition:** The columns of $AB$ are linear combinations of the columns of $A$. In fact, consider the $j$th column of $AB$.

**Row-column rule:** $(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$
**Definition.** The $n \times n$ (square) matrix

$$I_n = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix}$$

with 1’s down the diagonal and 0’s everywhere else is called the $n \times n$ **identity matrix**.

**Theorem 2.** Let $A$ be an $m \times n$ matrix, and let $B$ and $C$ be matrices of appropriate sizes. Then

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$
3. $(B + C)A = BA + CA$
4. $r(AB) = (rA)B = A(rB)$ for any scalar $r$
5. $I_mA = A = AI_n$

**Three warnings.**

1. $AB$ does not always equal $BA$.
2. $AB = AC$ does not necessarily imply that $B = C$.
3. $AB = 0$ does not necessarily imply that $A = 0$ or $B = 0$.

We will occasionally need to use the transpose of a matrix.

**Definition.** Given an $m \times n$ matrix $A$, its transpose $A^T$ is the $n \times m$ matrix such that

$$(A^T)_{ij} = A_{ji}.$$ 

**Example.** Consider

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6\pi \end{bmatrix}.$$
Theorem 3. Let $A$ and $B$ be matrices whose sizes are appropriate for the following sums and products. Then

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(rA)^T = rA^T$
4. $(AB)^T = B^T A^T$

Matrix inverses

Invertible square matrices are more well behaved than arbitrary square matrices.

**Definition.** Let $A$ be a square matrix such that there exists a square matrix $B$ such that either:

1. $AB = I$ or
2. $BA = I$.

Then we say that $A$ is *invertible* and that $B$ is the *inverse* of $A$.

**Examples.** Consider

$$A_1 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$