Eigenvectors and eigenvalues

**Definition.** Let $A$ be an $n \times n$ matrix. If $x$ is a nonzero vector such that

$$Ax = \lambda x$$

for some scalar $\lambda$, then $\lambda$ is an eigenvalue of $A$ and $x$ is an eigenvector associated to the eigenvalue $\lambda$.

Basic facts about eigenvectors

1. Any nonzero scalar multiple of an eigenvector is another eigenvector associated to the same eigenvalue.

2. The equation $Ax = \lambda x$ can be rewritten as

$$\left(A - \lambda I\right)x = 0.$$ 

If $\lambda$ is an eigenvalue for $A$, then $\text{Nul} \left(A - \lambda I\right)$ is the eigenspace associated to $\lambda$.

**Example.** Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ 1 & 1 & 0 \end{bmatrix}.$$ 

The numbers $\lambda = 1$ and $\lambda = 2$ are eigenvalues of $A$. What are the corresponding eigenspaces? The $\lambda = 1$ eigenspace:
The $\lambda = 2$ eigenspace:

**Theorem.** Let $\{v_1, \ldots, v_k\}$ be eigenvectors associated to distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. Then the set $\{v_1, \ldots, v_k\}$ is linearly independent.
How do we find the eigenvalues?

**Note:** The number $\lambda$ is an eigenvalue for the matrix $A$ if and only if the homogeneous system

$$(A - \lambda I) x = 0$$

has a nontrivial solution.

By the Invertible Matrix Theorem,

$$(A - \lambda I) x = 0$$

has a nontrivial solution if and only if the matrix $(A - \lambda I)$ is **not** invertible.

The number $\lambda$ is an eigenvalue for the matrix $A$ if and only if

$$\det(A - \lambda I) = 0.$$  

**Example.** Let

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$
Example. Let

\[ A = \begin{bmatrix} 3 & -2 \\ -1 & 0 \end{bmatrix}. \]

Example. Let

\[ A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ 1 & 1 & 0 \end{bmatrix}. \]
Example. Let
\[ A = \begin{bmatrix}
0 & 2 & 1 & -2 \\
-2 & 2 & -2 & 0 \\
-2 & 5 & 4 & -4 \\
3 & 6 & -6 & -6
\end{bmatrix}. \]