Definition. Suppose that $v_1, v_2, \ldots, v_p$ are vectors in $\mathbb{R}^n$. The set of all possible linear combinations of $v_1, v_2, \ldots, v_p$ is called the

$$\text{span}\{v_1, v_2, \ldots, v_p\}.$$ 

Note:

1. Every scalar multiple of each $v_k$ is in $\text{span}\{v_1, v_2, \ldots, v_p\}$.
2. The zero vector is always in the span of any set of vectors.
3. The $\text{span}\{v_1\}$ is the set of all scalar multiples of $v_1$.

Example. The set of all points $(x_1, x_2, x_3)$ in $\mathbb{R}^3$ that satisfy the equation

$$x_1 + x_2 + x_3 = 0$$

is a plane. How can we describe this plane using the vector operations?
The matrix-vector product $Ax$

Let $A$ be an $m \times n$ matrix and $x$ be a vector in $\mathbb{R}^n$. We can define the product $Ax$ as a linear combination of the vectors that come from the columns of $A$.

**Definition.** Let $A$ be an $m \times n$ matrix

\[
A = \begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
= \begin{bmatrix}
    A_1 \\
    A_2 \\
    \vdots \\
    A_n
\end{bmatrix},
\]

where $A_k$ is the $k$th column of $A$. Given

\[
x = \begin{bmatrix}
x_1 \\
\vdots \\
x_n
\end{bmatrix}
\]

in $\mathbb{R}^n$, we define the matrix-vector product $Ax$ to be the linear combination

\[x_1A_1 + x_2A_2 + \ldots + x_nA_n.\]

Note that $Ax$ is a vector in $\mathbb{R}^m$.

**Example.**

\[
\begin{bmatrix}
    3 & -8 \\
    -1 & 5 \\
    2 & -3
\end{bmatrix}
\begin{bmatrix}
    -4 \\
    2
\end{bmatrix}
= -4 \begin{bmatrix}
    3 \\
    -1 \\
    2
\end{bmatrix}
+ 2 \begin{bmatrix}
    -8 \\
    5 \\
    3
\end{bmatrix}
= \begin{bmatrix}
    (-4)(3) + (2)(-8) \\
    (-4)(-1) + (2)(5) \\
    (-4)(2) + (2)(3)
\end{bmatrix}
= \begin{bmatrix}
    -28 \\
    14 \\
    -2
\end{bmatrix}
\]
Remark. Given an $m \times n$ matrix $A$ and $x \in \mathbb{R}^n$, then the matrix equation

$$Ax = b$$

has the same solution set as the system of linear equations whose augmented matrix is

$$\left[ \begin{array}{c|c|c|c} A_1 & A_2 & \ldots & A_n & b \\ \hline A & b \end{array} \right].$$

Example. Which vectors $b$ are linear combinations of the vectors

$$A_1 = \begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -4 \\ -6 \\ 8 \end{bmatrix}?$$
Theorem. Let $A$ be an $m \times n$ matrix. Then the following three statements are equivalent:

1. For each $b$ in $\mathbb{R}^m$, the equation $Ax = b$ has at least one solution.

2. The columns of $A$ span $\mathbb{R}^m$.

3. The matrix $A$ has a pivot position in every row.

Warning: In this theorem, $A$ is a coefficient matrix. The three statements are not equivalent if $A$ is an augmented matrix.
Observation. Note that the $k$th entry in $Ax$ is

$$a_{k1}x_1 + a_{k2}x_2 + \ldots + a_{kn}x_n.$$ 

For example,

$$\begin{bmatrix} * & * \\ 5 & 6 \\ * & * \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} * \\ 5x_1 + 6x_2 \\ * \end{bmatrix}.$$ 

The expression

$$a_{k1}x_1 + a_{k2}x_2 + \ldots + a_{kn}x_n$$

is called the **dot product** of $[a_{k1} \ a_{k2} \ldots \ a_{kn}]$ and the vector $x$.

Theorem. Let $A$ be an $m \times n$ matrix. Then the matrix-vector product $Ax$ is “linear” in $x$. That is,

1. $A(u + v) = Au + Av$ for all $u$ and $v$ in $\mathbb{R}^n$, and

2. $A(cu) = cAu$ for all $u$ in $\mathbb{R}^n$ and all $c$ in $\mathbb{R}$.