Determinants

We start with a recursive definition of the determinant.

**Definition.** The determinant of a $1 \times 1$ matrix $[a_{11}]$ is $a_{11}$.

Now we define the determinant of an $n \times n$ matrix in terms of determinants of $(n-1) \times (n-1)$ matrices.

**Definition.** Given an $n \times n$ matrix $A$, the $ij$th minor $A_{ij}$ of $A$ is the $(n-1) \times (n-1)$ matrix obtained from $A$ by eliminating the $i$th row and $j$th column. The $ij$th cofactor of $A$ is

$$C_{ij} = (-1)^{i+j} \det A_{ij}.$$ 

**Example.** Consider the matrix

$$A = \begin{bmatrix} -1 & 4 & 7 \\ 3 & -2 & -2 \\ 4 & 0 & 2 \end{bmatrix}.$$
Definition/Theorem. If $A$ is an $n \times n$ matrix, the determinant of $A$ can be computed using cofactor expansion along the $i$th row by

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$$

or by cofactor expansion along the $j$th column by

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}.$$ 

Any row or any column yields the same result.

Note that we get the familiar formula

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$ 

Is there a way to define $\det A$ without recursion?
How do we go about computing det $A$?

One type of matrix is perfectly suited for cofactor expansion.

**Theorem.** If $A$ is a triangular matrix, then det $A$ is the product of its entries along the main diagonal.

In order to gain some insight into how we will compute determinants in general, let’s calculate the determinants of all elementary $3 \times 3$ matrices.