What is a vector space?

Definition. A vector space V is a set of objects that are called vectors along with two operations—vector addition and scalar multiplication. The vector sum $\mathbf{v}_1 + \mathbf{v}_2$ is always defined for any pair of vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in V, and given any scalar r in $\mathbb{R}$ and any vector $\mathbf{v}$ in V, the scalar multiple $r\mathbf{v}$ is a vector in V. Moreover, these two operations must satisfy the following eight properties:

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. There is a zero vector $\mathbf{0}$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
4. For each $\mathbf{u}$, there is a vector $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
6. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
7. $c(d\mathbf{u}) = (cd)\mathbf{u}$
8. $1\mathbf{u} = \mathbf{u}$

Example 1. The vector space $\mathbb{R}^n$. See p. 32 of our text for a discussion of the properties listed above.

Example 2. The vector space of all functions $f : \mathbb{R} \to \mathbb{R}$.

Example 3. The vector space $M_{m \times n}$ of all $m \times n$ matrices. (We assume that the entries of the matrices are real numbers, but a different vector space is obtained if one allows the entries to be complex numbers.)

Example 4. The vector space $\mathbb{P}$ of all polynomial functions $p : \mathbb{R} \to \mathbb{R}$.

Example 5. The vector space of all discrete-time signals.

Subspaces of vector spaces

Definition. A nonempty subset S of a vector space V is a *subspace* of V if

1. the zero vector $\mathbf{0}$ is in S,
2. (closure under vector addition) for each $\mathbf{v}_1$ and $\mathbf{v}_2$ in S, the vector sum $\mathbf{v}_1 + \mathbf{v}_2$ is in S, and
3. (closure under scalar multiplication) for each r in $\mathbb{R}$ and each $\mathbf{v}$ in S, the scalar multiple $r\mathbf{v}$ is in S.

Note. A subspace S of a vector space V is a vector space in its own right.

Example. Consider the line $x_2 = 3x_1$ in the vector space $\mathbb{R}^2$.

Example. Consider the line $x_2 = x_1 + 1$ in the vector space $\mathbb{R}^2$.

Example. Is $\mathbb{P}$ a subspace of the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$?

Example. Consider the subset $S = \text{Span}\{x, x^2\}$ within $\mathbb{P}$. Is S a subspace of $\mathbb{P}$?

Theorem. If $\mathbf{v}_1$, $\mathbf{v}_2$, $\ldots$, $\mathbf{v}_p$ are vectors in a vector space V, then $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is a subspace of V.