More on subspaces related to matrices

Recall that the null space of an $m \times n$-matrix is a subspace of $\mathbb{R}^n$.

**Application.** Any plane through the origin in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.

**Example.** Consider the matrix

$$A = \begin{bmatrix}
1 & -2 & 0 & 4 & 0 \\
0 & 0 & 1 & -9 & 0 \\
2 & -4 & 0 & 8 & 1
\end{bmatrix}.$$ 

Express the null space of $A$ as the span of as few vectors as possible.
The consistency of a system of linear equations can be viewed as a statement about the column space of the coefficient matrix.

**Fact.** The linear system $Ax = b$ is consistent if and only if $b$ is an element of the column space of $A$. 
Here is how Lay (p. 232) contrasts Nul $A$ and Col $A$ for an $m \times n$ matrix $A$:

**Nul $A$**

1. Nul $A$ is a subspace of $\mathbb{R}^n$.
2. Nul $A$ is implicitly defined; that is, you are given only a condition $(Ax = 0)$ that vectors in Nul $A$ must satisfy.
3. It takes time to find vectors in Nul $A$. Row operations on $[A \ 0]$ are required.
4. There is no obvious relation between Nul $A$ and the entries in $A$.
5. A typical vector $v$ in Nul $A$ has the property that $Av = 0$.
6. Given a specific vector $v$, it is easy to tell if $v$ is in Nul $A$. Just compute $Av$.
7. Nul $A = \{0\}$ if and only if the equation $Ax = 0$ has only the trivial solution.
8. Nul $A = \{0\}$ if and only if the linear transformation $x \mapsto Ax$ is one-to-one.

**Col $A$**

1. Col $A$ is a subspace of $\mathbb{R}^m$.
2. Col $A$ is explicitly defined; that is, you are told how to build vectors in Col $A$.
3. It is easy to find vectors in Col $A$. The columns of $A$ are displayed; others are formed from them.
4. There is an obvious relation between Col $A$ and the entries in $A$, since each column of $A$ is in Col $A$.
5. A typical vector $v$ in Col $A$ has the property that the equation $Ax = v$ is consistent.
6. Given a specific vector $v$, it may take time to tell if $v$ is in Col $A$. Row operations on $[A \ v]$ are required.
7. Col $A = \mathbb{R}^m$ if and only if the equation $Ax = b$ has a solution for every $b$ in $\mathbb{R}^m$.
8. Col $A = \mathbb{R}^m$ if and only if the linear transformation $x \mapsto Ax$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$. 
Item 8 in both lists suggest two subspaces that are intimately connected with any linear transformation from one vector space to another.

**Definition.** A transformation \( L : V_1 \to V_2 \) from a vector space \( V_1 \) to a vector space \( V_2 \) is linear if

1. \( L(v_1 + v_2) = L(v_1) + L(v_2) \) for all vectors \( v_1 \) and \( v_2 \) in \( V_1 \), and
2. \( L(rv) = rL(v) \) for all \( v \) in \( V_1 \) and all \( r \) in \( \mathbb{R} \).

**Example.** Let \( V_1 \) be the vector space of all continuously differentiable functions \( f : \mathbb{R} \to \mathbb{R} \) and let \( V_2 \) be the vector space of all continuous functions \( f : \mathbb{R} \to \mathbb{R} \). The operation of differentiation is a linear transformation from \( V_1 \) to \( V_2 \). That is, the transformation \( D : V_1 \to V_2 \) given by

\[
D(f) = f'
\]

is a linear transformation.

Associated to any linear transformation are two important subspaces.

**Definition.** The kernel of \( L : V_1 \to V_2 \) is the subset of \( V_1 \) given by

\[
\{v_1 \mid L(v_1) = 0\}.
\]

The range of \( L \) is the subset of \( V_2 \) given by

\[
\{v_2 \mid L(v_1) = v_2 \text{ for some } v_1 \text{ in } V_1\}.
\]

**Fact.** Both the kernel and the range of a linear transformation are subspaces. The kernel is a subspace of \( V_1 \), and the range is a subspace of \( V_2 \).
For a matrix transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ determined by the matrix $A$, its range is $\text{Col } A$, and its kernel is $\text{Nul } A$.

**Example.** What are the kernel and range of the transformation $p : \mathbb{R}^3 \to \mathbb{R}^3$ determined by the matrix

$$
\frac{1}{3} \begin{bmatrix}
2 & -1 & 1 \\
-1 & 2 & 1 \\
1 & 1 & 2
\end{bmatrix}
$$

(This example was first introduced on September 25.)
Example. What are the kernel and the range of the differentiation transformation $D$ mentioned above?