Coordinates relative to a basis

A basis for a vector space produces a coordinate system for that space.

**Theorem.** (Unique Representation Theorem) Let $B = \{b_1, \ldots, b_n\}$ be a basis for a vector space $V$. Then every vector $v$ in $V$ can be represented uniquely as

$$v = c_1 b_1 + \ldots + c_n b_n.$$ 

The scalars $c_1, \ldots, c_n$ are called the coordinates of $v$ relative to the basis $B$.

**Example.** Consider the spanning set $\{x^3 + 1, x, x^2, x^2 - x, 4, x^3\}$ for the vector space $\mathbb{P}_3$. There are infinitely many ways to write a given element of $\mathbb{P}_3$ as a linear combination of these vectors. For example, consider the polynomial $2x^3 - x^2$. It can be written as

$$(-1)x^2 + 2x^3.$$ 

It can also be written as

$$2(x^3 + 1) + (-1)x + (-1)(x^2 - x) + (-\frac{1}{2})(4).$$ 

Because this spanning set is not linearly independent, there is no unique representation of $2x^3 - x^2$ as a linear combination of the vectors.

Last class we produced a basis of $\mathbb{P}_3$ from this spanning set using the casting-out procedure. The basis is $\{x^3 + 1, x, x^2, 4\}$. What are the coordinates of $2x^3 - x^2$ relative to this basis?

Why are coordinates relative to a given basis unique?
The same vector has different coordinates relative to different bases.

**Example.** Consider the vector

\[ \mathbf{x} = \begin{bmatrix} -1 \\ -3 \end{bmatrix} \]

in \( \mathbb{R}^2 \). What are its coordinates relative to the standard basis \( \{ \mathbf{e}_1, \mathbf{e}_2 \} \) and what are its coordinates relative to the basis

\[ B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}. \]

**Notation.** Given the representation \( \mathbf{v} = c_1 \mathbf{b}_1 + \ldots + c_n \mathbf{b}_n \) relative to the basis \( B = \{ \mathbf{b}_1, \ldots, \mathbf{b}_n \} \), then the coordinates can be viewed as a vector in \( \mathbb{R}^n \). This vector is denoted

\[ [\mathbf{v}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}. \]

**Example.** What is the coordinate vector for \( 2x^3 - x^2 \) relative to the basis \( B = \{ x^3 + 1, x, x^2, 4 \} \) of \( \mathbb{P}_3 \)?
Change of coordinates matrix

If \( B = \{b_1, \ldots, b_n\} \) is a basis of \( \mathbb{R}^n \), then the \( B \)-coordinates of a vector \( x \) are related to the standard coordinates by the equation

\[
x = c_1 b_1 + \ldots + c_n b_n.
\]

This equation can be rewritten in terms of matrix multiplication as

\[
x = P_B \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}
= P_B [x]_B
\]

where \( P_B \) is the matrix

\[
P_B = \begin{bmatrix} b_1 & b_2 & \ldots & b_n \end{bmatrix}.
\]

Since \( P_B \) is invertible, we also have \([x]_B = (P_B)^{-1} x\).

**Example.** We can double check our computation of the \( B \)-coordinates for the vector

\[
x = \begin{bmatrix} -1 \\ -3 \end{bmatrix}
\]

in \( \mathbb{R}^2 \) relative to the basis

\[
B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}
\]

using these equations.
For any vector space $V$ with basis $B = \{b_1, \ldots, b_n\}$, the $B$-coordinates define a nice linear transformation from $V$ onto $\mathbb{R}^n$. The map is defined by

$$v \mapsto [v]_B.$$ 

**Theorem.** The coordinate transformation $v \mapsto [v]_B$ is a one-to-one linear transformation that maps $V$ onto $\mathbb{R}^n$.

**Definition.** A one-to-one linear transformation that maps $V$ onto $W$ is called an isomorphism.

From the vector space point of view, two isomorphic vector spaces have the same structure.

**Example.** For what $n$ is $\mathbb{R}^n$ isomorphic to $\mathbb{P}_3$?