The Invertible Matrix Theorem

**Theorem.** Let $A$ be an $n \times n$ matrix. Then the following twelve statements are equivalent:

(a) $A$ is an invertible matrix.

(b) $A$ is row equivalent to the identity matrix.

(c) $A$ has $n$ pivot positions

(d) The equation $Ax = 0$ has no nontrivial solutions.

(e) The columns of $A$ are linearly independent.

(f) The linear transformation $T(x) = Ax$ is one-to-one.

(g) The equation $Ax = b$ has at least one solution for each $b \in \mathbb{R}^n$.

(h) The columns of $A$ span $\mathbb{R}^n$.

(i) The linear transformation $T(x) = Ax$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

(j) There is an $n \times n$ matrix $C$ such that $CA = I$.

(k) There is an $n \times n$ matrix $D$ such that $AD = I$.

(l) $A^T$ is an invertible matrix.
Computer graphics

Homogeneous coordinates are useful when we want to do computer graphics with matrices.

**Definition.** A point \((x, y)\) in \(\mathbb{R}^2\) can be represented by the point \((x, y, 1)\) in \(\mathbb{R}^3\). The coordinates \((x, y, 1)\) are called the homogeneous coordinates of the point \((x, y)\).

Homogeneous coordinates are useful because translation in \(\mathbb{R}^2\) can be represented by a linear transformation in \(\mathbb{R}^3\).

**Fact 1.** A translation by \((h, k)\) in \(\mathbb{R}^2\) can be obtained by matrix multiplication of homogeneous coordinates. That is,

\[
\begin{bmatrix}
1 & 0 & h \\
0 & 1 & k \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
1
\end{bmatrix} =
\begin{bmatrix}
x + h \\
y + k \\
1
\end{bmatrix}.
\]

**Fact 2.** Any linear transformation \(\mathbb{R}^2 \to \mathbb{R}^2\) can be represented as a transformation of homogeneous coordinates by matrix multiplication. In particular, if the transformation is represented by the matrix

\[
A = \begin{bmatrix}
a & b \\
c & d
\end{bmatrix},
\]

then the corresponding matrix for homogeneous coordinates is

\[
\begin{bmatrix}
a & b & 0 \\
c & d & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

**Example.** What is the matrix that represents rotation by 45° in terms of homogeneous coordinates?