A little more on orthogonal complements

**Definition.** Given a subspace $S$ of $\mathbb{R}^n$, its orthogonal complement $S^\perp$ is the set

$$\{ v \mid v \cdot w = 0 \text{ for all } w \in S \}.$$  

**Theorem.** Let $S$ be a subspace of $\mathbb{R}^n$ and let $S^\perp$ be its orthogonal complement. Then

1. $S^\perp$ is a subspace of $\mathbb{R}^n$,
2. $\dim(S^\perp) = n - \dim(S)$,
3. $(S^\perp)^\perp = S$, and
4. every vector $v$ in $\mathbb{R}^n$ can be written uniquely as $v = v_1 + v_2$, where $v_1$ is in $S$ and $v_2$ is in $S^\perp$.

It helps to have a little more theory before we can verify properties 2, 3, and 4, but we can verify property 1 directly from the definition.

Orthogonal sets

**Definition.** A set of vectors $\{v_1, v_2, \ldots, v_k\}$ is an orthogonal set if $v_i \cdot v_j = 0$ for all $i \neq j$.

**Example 1.** Consider the vectors

$$v_1 = \begin{bmatrix} 2 \\ -1 \\ 4 \\ 5 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \quad \text{and} \quad v_3 = \begin{bmatrix} 0 \\ 3 \\ 2 \\ -1 \end{bmatrix}.$$
Theorem. Suppose that \( \{v_1, v_2, \ldots, v_k\} \) is an orthogonal set of nonzero vectors.

1. If \( u = c_1v_1 + c_2v_2 + \ldots + c_kv_k \), then the weights \( c_i \) are given by \( c_i = \frac{u \cdot v_i}{v_i \cdot v_i} \).

2. The set \( \{v_1, v_2, \ldots, v_k\} \) is linearly independent.
Example. Using the orthogonal set \( \{v_1, v_2, v_3\} \) in Example 1, apply this theorem to the vector

\[
v = \begin{bmatrix} -45 \\ -4 \\ 3 \\ 1 \end{bmatrix}.
\]
Orthonormal sets

**Definition.** A set of vectors \( \{v_1, v_2, \ldots, v_k\} \) is orthonormal if it is orthogonal and \( v_i \cdot v_i = 1 \) for all \( i \).

**Example.** Consider the vectors

\[
\begin{align*}
v_1 &= \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, &
v_2 &= \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, &
v_3 &= \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}.
\end{align*}
\]
We can use matrices to express the fact that a set is orthonormal.