A little review of orthogonal and orthonormal sets

**Definition.** A set of vectors \( \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \) is an orthogonal set if \( \mathbf{v}_i \cdot \mathbf{v}_j = 0 \) for all \( i \neq j \).

**Theorem.** Suppose that \( \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \) is an orthogonal set of nonzero vectors.

1. If \( \mathbf{u} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \ldots + c_k \mathbf{v}_k \), then the weights \( c_i \) are given by \( c_i = \frac{\mathbf{u} \cdot \mathbf{v}_i}{\mathbf{v}_i \cdot \mathbf{v}_i} \).

2. The set \( \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \) is linearly independent.

**Definition.** A set of vectors \( \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \) is orthonormal if it is orthogonal and \( \mathbf{v}_i \cdot \mathbf{v}_i = 1 \) for all \( i \).

**Example.** The three vectors

\[
\mathbf{v}_1 = \begin{bmatrix} \frac{2}{3} \\ \frac{-1}{3} \\ \frac{2}{3} \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} \frac{-1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} \frac{2}{3} \\ \frac{-1}{3} \\ \frac{-1}{3} \end{bmatrix}
\]

form an orthonormal set in \( \mathbb{R}^3 \).

We can use matrices to express the fact that a set is orthogonal or orthonormal.
**Theorem.** Let $A$ be an $n \times n$ matrix. The following three conditions are equivalent.

1. $A^T = A^{-1}$

2. The columns of $A$ form an orthonormal basis of $\mathbb{R}^n$.

3. The rows of $A$ form an orthonormal basis of $\mathbb{R}^n$.

**Definition.** Whenever a matrix satisfies the above theorem, it is said to be an orthogonal matrix.

**Example.** We can use the orthonormal basis of $\mathbb{R}^3$ given above to produce an orthogonal matrix.

Why are orthogonal matrices special?
Orthogonal projection

How do we project a vector $v$ onto a subspace $W$?

**Theorem.** (Orthogonal Decomposition Theorem)

1. Each vector $v$ in $\mathbb{R}^n$ can be written uniquely as

$$v = w + w^\perp,$$

where $w$ is in $W$ and $w^\perp$ is in $W^\perp$.

2. Given an orthogonal basis $\{w_1, \ldots, w_k\}$ of $W$, then

$$w = \left( \frac{v \cdot w_1}{w_1 \cdot w_1} \right) w_1 + \ldots + \left( \frac{v \cdot w_k}{w_k \cdot w_k} \right) w_k$$

and $w^\perp = v - w$. 

\[ 
\begin{array}{c}
\text{Orthogonal projection} \\
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\textbf{Theorem.} \ (\text{Orthogonal Decomposition Theorem}) \\
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\quad v = w + w^\perp, \\
\text{where } w \text{ is in } W \text{ and } w^\perp \text{ is in } W^\perp. \\
\text{2. Given an orthogonal basis } \{w_1, \ldots, w_k\} \text{ of } W, \text{ then} \\
\quad w = \left( \frac{v \cdot w_1}{w_1 \cdot w_1} \right) w_1 + \ldots + \left( \frac{v \cdot w_k}{w_k \cdot w_k} \right) w_k \\
\text{and } w^\perp = v - w. \\
\end{array} 
\]
Why is the Orthogonal Decomposition Theorem true?
Important consequence: If we want to find the distance of a vector $v$ to a subspace $W$, then we compute

$$||w^\perp|| = ||v - w||.$$ 

**Example.** Find the point closest to

$$v = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix}$$

in the subspace $W$ spanned by the two vectors

$$w_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix} \quad \text{and} \quad w_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}.$$
Theorem. If \(\{u_1, \ldots, u_k\}\) is an orthonormal basis for a subspace \(W\), then

\[ w = (v \cdot u_1)u_1 + \ldots + (v \cdot u_k)u_k. \]

If

\[ U = \begin{bmatrix} u_1 & u_2 & \ldots & u_k \end{bmatrix}, \]

then \(w = UU^T v\).