More on orthogonal projection
Recall the Orthogonal Decomposition Theorem from last class.

**Theorem.** (Orthogonal Decomposition Theorem)

1. Each vector \( \mathbf{v} \) in \( \mathbb{R}^n \) can be written uniquely as
   \[
   \mathbf{v} = \mathbf{w} + \mathbf{w}^\perp,
   \]
   where \( \mathbf{w} \) is in \( W \) and \( \mathbf{w}^\perp \) is in \( W^\perp \).

2. Given an orthogonal basis \( \{\mathbf{w}_1, \ldots, \mathbf{w}_k\} \) of \( W \), then
   \[
   \text{proj}_W \mathbf{v} \equiv \mathbf{w} = \left( \frac{\mathbf{v} \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1} \right) \mathbf{w}_1 + \ldots + \left( \frac{\mathbf{v} \cdot \mathbf{w}_k}{\mathbf{w}_k \cdot \mathbf{w}_k} \right) \mathbf{w}_k
   \]
   and \( \mathbf{w}^\perp = \mathbf{v} - \mathbf{w} \).

Note: Since the two vectors \( \mathbf{w} \) and \( \mathbf{w}^\perp \) are unique, they do not depend on the orthogonal basis of \( W \) that we use to compute them.

Important consequence: If we want to find the distance of a vector \( \mathbf{v} \) to a subspace \( W \), then we compute
   \[
   ||\mathbf{w}^\perp|| = ||\mathbf{v} - \text{proj}_W \mathbf{v}||.
   \]

**Example.** Find the point closest to
\[
\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix}
\]
in the subspace \( W \) spanned by the two vectors
\[
\mathbf{w}_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{w}_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}.
\]
Theorem. If \( \{u_1, \ldots, u_k\} \) is an orthonormal basis for a subspace \( W \), then

\[
\text{proj}_W v = (v \cdot u_1)u_1 + \ldots + (v \cdot u_k)u_k.
\]

If

\[
U = \begin{bmatrix}
    u_1 & u_2 & \ldots & u_k
\end{bmatrix},
\]

then \( \text{proj}_W v = UU^T v \).
Example. Let’s redo the previous example using the projection matrix. Let

\[
U = \begin{bmatrix}
-\frac{1}{\sqrt{10}} & -\frac{4}{\sqrt{26}} \\
\frac{2}{\sqrt{10}} & \frac{1}{\sqrt{26}} \\
-\frac{1}{\sqrt{10}} & 0 \\
2 & 3
\end{bmatrix}
\]
The Gram-Schmidt Process

This procedure produces an orthogonal (or orthonormal) basis from a basis \( \{x_1, \ldots, x_p\} \) of a subspace \( W \). It is an inductive procedure.

We work with the subspaces

\[ S_l = \text{Span}\{x_1, \ldots, x_l\}. \]

The orthogonal basis for \( W \) based on this procedure applied to this basis is denoted \( \{v_1, \ldots, v_l\} \).

1. Let \( v_1 = x_1 \).

2. Let \( v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1 \).

3. Let \( v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2 \).

etc.
Example. Apply the Gram-Schmidt process to the basis

\[ \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{x}_3 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}. \]
Example. Let’s use these ideas to find the projection matrix $P$ for orthogonal projection onto the plane $x_1 + x_2 - x_3 = 0$ in $\mathbb{R}^3$.

What are the eigenvalues and eigenspaces of $P$? (No computation required)