Projection matrices

We discussed projection matrices briefly last class. In particular, we discussed the following theorem.

**Theorem.** Let \( \{u_1, \ldots, u_k\} \) be an orthonormal basis for a subspace \( W \) of \( \mathbb{R}^n \). Form the \( n \times k \) matrix

\[
U = \begin{bmatrix}
u_1 & u_2 & \ldots & u_k
\end{bmatrix}.
\]

Then \( \text{proj}_W v = UU^T v \).

The matrix \( P = UU^T \) is called the *projection matrix* for the subspace \( W \). It does not depend on the choice of orthonormal basis.

**Example.** Let’s compute the projection matrix \( P \) for orthogonal projection onto the plane \( x_1 + x_2 - x_3 = 0 \) in \( \mathbb{R}^3 \).
What are the eigenvalues and eigenspaces of $P$? (No computation required)

What if we do not start with an orthonormal basis of $W$?

**Theorem.** Let $\{a_1, \ldots, a_k\}$ be any basis for a subspace $W$ of $\mathbb{R}^n$. Form the $n \times k$ matrix

$$A = \begin{bmatrix} a_1 & a_2 & \ldots & a_k \end{bmatrix}.$$ 

Then the projection matrix for $W$ is $A(A^TA)^{-1}A^T$.

A proof of this fact is posted on the course web site.

Note that any projection matrix $P$ satisfies the two properties

1. $P^2 = P$, and
2. $P$ is symmetric.

It is also true that any matrix that satisfies these two properties is the projection matrix for some subspace of $\mathbb{R}^n$.  

2
Least squares approximation

Suppose we have data points \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\) and we want to “fit” them to a line.

How do we find the equation \(y = mx + b\) of the line?

Form the matrices

\[
Y_d = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}.
\]

If all of the points are on the line \(y = mx + b\), we have

\[
Y_d = X \begin{bmatrix} b \\ m \end{bmatrix},
\]

and we are done.

If not, we consider all \(Y\) of the form

\[
Y = X \begin{bmatrix} b \\ m \end{bmatrix}
\]

for all possible \(b\) and \(m\), and we look for the one that is “closest” to \(Y_d\). In other words, we minimize

\[
||Y_d - X \begin{bmatrix} b \\ m \end{bmatrix}||
\]

as we vary \(b\) and \(m\).
Note also that we sweep out the column space of $\mathbf{X}$ as we vary $b$ and $m$. Therefore, the minimum is attained when 

$$
\mathbf{X} \begin{bmatrix} b \\ m \end{bmatrix}
$$

is the projection of $\mathbf{Y}_d$ onto the column space of $\mathbf{X}$.

Using the formula given earlier for the projection matrix, we have

$$
\mathbf{X} \begin{bmatrix} b \\ m \end{bmatrix} = \mathbf{X}(\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{Y}_d.
$$

Although $\mathbf{X}$ is not square and therefore not invertible, it has rank 2, and consequently, the transformation induced by $\mathbf{X}$ is one-to-one. We can cancel $\mathbf{X}$ from the left on both sides, and we obtain

$$
\begin{bmatrix} b \\ m \end{bmatrix} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{Y}_d.
$$

**Example.** Suppose that a company had profits of $500,000 in year 1, $1,000,000 in year 2, and $2,000,000 in year 5. Model its profits with a least-squares linear model.
Example. Here are relative growth rates for the U.S. population from 1800 to 1990.

<table>
<thead>
<tr>
<th>Year</th>
<th>U.S. Population</th>
<th>Rel Growth Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1800</td>
<td>5.3</td>
<td>0.03113</td>
</tr>
<tr>
<td>1810</td>
<td>7.2</td>
<td>0.02986</td>
</tr>
<tr>
<td>1820</td>
<td>9.6</td>
<td>0.02500</td>
</tr>
<tr>
<td>1830</td>
<td>12</td>
<td>0.03083</td>
</tr>
<tr>
<td>1840</td>
<td>17</td>
<td>0.03235</td>
</tr>
<tr>
<td>1850</td>
<td>23</td>
<td>0.03043</td>
</tr>
<tr>
<td>1860</td>
<td>31</td>
<td>0.02419</td>
</tr>
<tr>
<td>1870</td>
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</tr>
<tr>
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<td>50</td>
<td>0.02400</td>
</tr>
<tr>
<td>1890</td>
<td>62</td>
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<tr>
<td>1900</td>
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<tr>
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<td>91</td>
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<td>1920</td>
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<tr>
<td>1930</td>
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<td>0.01066</td>
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<tr>
<td>1940</td>
<td>131</td>
<td>0.01107</td>
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<tr>
<td>1950</td>
<td>151</td>
<td>0.01589</td>
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<tr>
<td>1960</td>
<td>179</td>
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<tr>
<td>1990</td>
<td>249</td>
<td>0.01094</td>
</tr>
</tbody>
</table>

Here's a graph of these relative growth rates versus population: