More on quadratic forms

Last class we saw that the Spectral Theorem for symmetric matrices can be used to eliminate “mixed” terms in a quadratic form. In fact, the Spectral Theorem implies the Principal Axes Theorem for quadratic forms.

Today we examine a few consequences.

Quadratic forms are classified according to the signs of their values.

**Definition.** A quadratic form $Q$ is:

1. positive definite if $Q(x) > 0$ for all $x \neq 0$,
2. negative definite if $Q(x) < 0$ for all $x \neq 0$, and
3. indefinite if $Q(x)$ assumes both positive and negative values.

There are also definitions of positive/negative semidefinite quadratic forms given in the textbook.

**Example.** Suppose that the quadratic form $Q : \mathbb{R}^2 \to \mathbb{R}$ is positive definite. What does the graph of

$$x_3 = Q(x_1, x_2)$$

look like?
Theorem. Let $A$ be a symmetric matrix. Then the quadratic form $x^T A x$ is:

1. positive definite if and only if all of the eigenvalues of $A$ are positive.
2. negative definite if and only if all of the eigenvalues of $A$ are negative.
3. indefinite if and only if $A$ has both positive and negative eigenvalues.
I would end the semester by returning to a topic that we discussed early in the semester. On September 20, we discussed the structure of solution sets to nonhomogeneous linear equations. Here’s a repeat of a theorem from that class:

**Theorem.** Let \( p \) be one solution to the nonhomogeneous equation \( Ax = b \) and let \( S \) be the set of all solutions to the associated homogeneous equation

\[
Ax = 0.
\]

Then the solution set of \( Ax = b \) consists of all vectors in the set \( p + S \).

In Exercise 23 of Section 6.3, we proved that there is a unique \( p \in \text{Row } A \) that is a solution to a consistent nonhomogeneous system. Now I would like to describe this result using diagrams advocated by Professor Gilbert Strang at MIT.