1. (10 points) Consider the subset H of $\mathbb{R}^4$ given by

$$\left\{ \begin{bmatrix} a-2b-2c+3d \\ 2a-4b-3c+9d \\ a-2b-c+6d \\ c+3d \end{bmatrix} \ : \ a,b,c,d \text{ in } \mathbb{R} \right\}.$$

(a) Explain why H is a subspace of $\mathbb{R}^4$.

$$H = \left\{ a\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} -2 \\ -4 \\ -2 \\ 0 \end{bmatrix} + c\begin{bmatrix} -2 \\ -3 \\ -1 \\ 1 \end{bmatrix} + d\begin{bmatrix} 3 \\ 9 \\ 6 \\ 3 \end{bmatrix} \ \middle| \ a,b,c,d \text{ in } \mathbb{R} \right\}.$$

$$\Rightarrow H = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ 6 \\ 3 \end{bmatrix} \right\}.$$

(b) Find a basis for H and calculate its dimension.

$$H = \text{Col}\,A \text{ where } A = \begin{bmatrix} 1 & -2 & -2 & 3 \\ 2 & -4 & -3 & 9 \\ 1 & -2 & -1 & 6 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

Then

$$A \sim \begin{bmatrix} 1 & -2 & -2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Columns 1 and 3 are pivot columns.

$$H = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ -1 \\ 1 \end{bmatrix} \right\} \qquad \dim H = 2.$$

2. (14 points) Consider the system

$$\begin{aligned} x_1 + 3x_4 &= 4 \\ x_2 + x_3 - 3x_4 &= 2 \\ -x_1 - 3x_4 + x_5 &= -1 \\ 3x_1 + 9x_4 - 2x_5 &= 6 \end{aligned}$$

of four equations in five unknowns.

(a) Express its solution set in parametric vector form.

Augmented matrix:

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 3 & 0 & 4 \\ 0 & 1 & 1 & -3 & 0 & 2 \\ -1 & 0 & 0 & -3 & 1 & -1 \\ 3 & 0 & 0 & 9 & -2 & 6 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 3 & 0 & 4 \\ 0 & 1 & 1 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & -2 & -6 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 3 & 0 & 4 \\ 0 & 1 & 1 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The system is consistent.

x_3 and x_4 are free variables. To obtain a particular solution to the NH system, let $x_3 = x_4 = 0$. Then

$x_5 = 3$, $x_4 = 0$, $x_3 = 0$, $x_2 = 2$, $x_1 = 4$.

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Problem 2 (continued):

To obtain the general solution of the associated homogeneous solution, we let

$x_3 = 1,\ x_4 = 0 \Rightarrow x_5 = 0,\ x_2 = -1,\ x_1 = 0.$

$x_3 = 0,\ x_4 = 1 \Rightarrow x_5 = 0,\ x_2 = 3,\ x_1 = -3.$

Solution set of the nonhomogeneous system:

$$\begin{bmatrix} 4 \\ 2 \\ 0 \\ 0 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

(b) Is it possible to change the constants on the right-hand side of the system so that the new system is inconsistent? In order to receive any credit, you must justify your answer.

Yes. The coefficient matrix A gives a transformation $A: \mathbb{R}^5 \to \mathbb{R}^4$ with 3 pivot columns. Therefore, $\dim \operatorname{Col} A = 3$. Since $\dim \mathbb{R}^4 = 4$, there are infinitely many b such that $Ax = b$ is inconsistent.

3. (16 points) Let

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & -4 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{bmatrix}.$$

(a) Calculate the characteristic polynomial and eigenvalues of $\mathbf{A}$.

$$\det(A-\lambda I) = \det \begin{bmatrix} 3-\lambda & 0 & -4 \\ 0 & -2-\lambda & -1 \\ 0 & 2 & 1-\lambda \end{bmatrix}$$

Cofactor expansion down the first column:

$$\det(A-\lambda I) = (3-\lambda) \det \begin{bmatrix} -2-\lambda & -1 \\ 2 & 1-\lambda \end{bmatrix}$$

$$= (3-\lambda)((-2-\lambda)(1-\lambda)+2)$$

$$= (3-\lambda)((\lambda+2)(\lambda-1)+2)$$

$$= (3-\lambda)(\lambda^2+\lambda-\cancel{2}+\cancel{2})$$

$$= (3-\lambda)(\lambda^2+\lambda)$$

$$= (3-\lambda)(\lambda)(\lambda+1)$$

eigenvalues: $\lambda = 3, 0, -1$.

Problem 3 (continued): Here is the matrix **A** again:

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & -4 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{bmatrix}.$$

(b) Diagonalize **A**. In other words, find an invertible matrix **P** and a diagonal matrix **D** such that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$. **(You do not need to calculate $\mathbf{P}^{-1}$.)**

From the matrix, we can see that $Ae_1 = 3e_1$.

$\Rightarrow$ $\lambda = 3$ eigenspace is Span$\{e_1\}$.

$\lambda = 0$ eigenspace is nul A:

$$\begin{bmatrix} 3 & 0 & -4 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & -4 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow x_3 \text{ is free.}$$

$x_3 = 1 \Rightarrow x_2 = -1/2,\ x_1 = 4/3$.

eigenspace = Span$\left\{ \begin{bmatrix} 8 \\ -3 \\ 6 \end{bmatrix} \right\}$

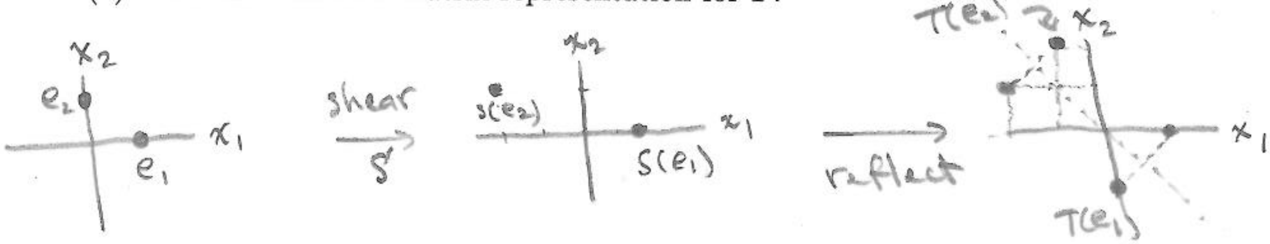
$\lambda = -1$ eigenspace is nul $\begin{bmatrix} 4 & 0 & -4 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{bmatrix} \sim$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \lambda = -1 \text{ espace Span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

Then $P = \begin{pmatrix} 1 & 8 & 1 \\ 0 & -3 & -1 \\ 0 & 6 & 1 \end{pmatrix}$ and $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

4. (8 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that first performs a horizontal shear that transforms $\mathbf{e}_2$ to $\mathbf{e}_2 - 2\mathbf{e}_1$ (leaving $\mathbf{e}_1$ unchanged) and then reflects points through the line $x_2 = -x_1$.

   (a) Find the standard matrix representation for T.



$$T(e_1) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$T(e_2) = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \Rightarrow T = \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}$$

   (b) Show that T is invertible and find a formula for T^{-1}.

$\det T = -1 \Rightarrow T$ is invertible.

$$T^{-1} = \frac{1}{-1}\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -1 & 0 \end{bmatrix}$$

5. (18 points) Let

$$\mathbf{A} = \begin{bmatrix} 0 & -1 & -2 \\ 0 & 1 & 0 \\ 3 & 0 & 6 \end{bmatrix}.$$

(a) Compute $\mathbf{A}^{-1}$. You may use your calculator to double check your answer, but you will not get any credit unless you show enough work so that I can be sure that you can do this problem without your calculator.

$$\left[\begin{array}{ccc|ccc} 0 & -1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 3 & 0 & 6 & 0 & 0 & 1 \end{array}\right] \sim_{R_1 \leftrightarrow R_3}$$

$$\left[\begin{array}{ccc|ccc} 3 & 0 & 6 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & -2 & 1 & 0 & 0 \end{array}\right] \sim_{R_3 \to R_3 + R_2}$$

(See Part c) $\rightarrow$ $$\left[\begin{array}{ccc|ccc} 3 & 0 & 6 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 & 1 & 0 \end{array}\right] \sim_{R_1 \to R_1 + 3R_3}$$

$$\left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 3 & 3 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 & 1 & 0 \end{array}\right] \sim \begin{array}{l} R_1 \to \frac{1}{3}R_1 \\ R_3 \to -\frac{1}{2}R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1/3 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1/2 & -1/2 & 0 \end{array}\right] \Rightarrow$$

$$A^{-1} = \begin{bmatrix} 1 & 1 & 1/3 \\ 0 & 1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}$$

Problem 5. (continued)

(b) Write $\mathbf{A}^{-1}$ as a product of elementary matrices. You do not need to multiply the elementary matrices together when you write $\mathbf{A}^{-1}$ as a product.

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_4 = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$$

$$A^{-1} = E_5 E_4 E_3 E_2 E_1$$

(c) Using the calculations that you have already made, determine the value of det $\mathbf{A}$. At what point in those calculations were you sure about this value?

See step indicated in Part A. At that point, the matrix is upper triangular. Its determinant is -6. The first row op changes the determinant by a -1. The second row op does not change the determinant. $\Rightarrow \det A = +6$.

6. (10 points) The trace of a matrix is the sum of its entries along the diagonal. For example, the trace of the 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is $a + d$. Consider the subset S of the vector space $M_{2\times 2}$ of all 2×2 matrices that consists of all matrices whose trace is zero.

(a) Show that S is a vector subspace of $M_{2\times 2}$.

(1) (not necessary) S contains zero vector $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
Note that the trace of $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is zero.

(2) closure under vector addition:
Let $A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & -a_2 \end{bmatrix}$.

Then $A_1 + A_2 = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & -(a_1+a_2) \end{bmatrix}$ has trace zero.

(3) closure under scalar multiplication:
Let $A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ and $r \in \mathbb{R}$.

Then $rA = \begin{bmatrix} ra & rb \\ rc & -ra \end{bmatrix}$ has zero trace.

Problem 6 (continued):

(b) Determine a basis for S. What is the dimension of S? Justify that your answer is a basis.

basis: $\left\{ A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$

$\Rightarrow \dim S = 3.$

To verify that $\{A_1, A_2, A_3\}$ is a basis:

1 linearly independent: let $r_1, r_2, r_3 \in \mathbb{R}$ such that $r_1 A_1 + r_2 A_2 + r_3 A_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

But $r_1 A_1 + r_2 A_2 + r_3 A_3 = \begin{bmatrix} r_1 & r_2 \\ r_3 & -r_1 \end{bmatrix}$.

$\Rightarrow r_1 = r_2 = r_3 = 0$

$\Rightarrow$ no nontrivial dependence relations.

2 $S = \text{Span}\{A_1, A_2, A_3\}$ Consider an arbitrary element $\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ of S.

Then $\begin{bmatrix} a & b \\ c & -a \end{bmatrix} = aA_1 + bA_2 + cA_3$.

1 and 2 $\Rightarrow \{A_1, A_2, A_3\}$ is a basis for S.

7. (24 points) Are the following statements true or false? You must justify your answers to receive any credit.

(a) A homogeneous linear system of three equations in five variables has at least two free variables.

True. $Ax = 0$ with A 3×5 matrix.

A can have at most 3 pivot positions

$\Rightarrow$ at most 3 pivot columns

$\Rightarrow$ at least two free variables.

(b) The columns of any 3×4 matrix are linearly dependent.

True. The columns of a 3×4 matrix make 4 vectors in $\mathbb{R}^3$. Since $\mathbb{R}^3$ is three-dimensional, any set of four vectors is linearly dependent.

(c) Let T be a linear transformation. If the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, then the set $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly independent.

False. Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be $T(x) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

and $\{v_1, v_2, v_3\}$ be $\{e_1, e_2, e_3\}$.

Then $\{T(v_1), T(v_2), T(v_3)\} = \{0, e_2, e_3\}$, which is linearly dependent.

Problem 7 (continued):

(d) Let **A** be an $m \times n$ matrix. The subspace Col **A** is the set of all vectors of the form $\mathbf{Ax}$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

True. Col A = span of the columns of A.
If $A = [a_1 | a_2 | \cdots | a_n]$, then Col A = all vectors of the form $x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$ for all choices of weights $x_1, \ldots, x_n$ in $\mathbb{R}$. Let $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$. Then $Ax = x_1 a_1 + \cdots + x_n a_n$.

(e) If a finite set S of nonzero vectors spans a vector space V, then some subset of S is a basis for V.

True. Apply the casting out procedure to S. The result will be a basis consisting entirely of vectors in S.

(f) A square matrix **A** is not invertible if and only if 0 is an eigenvalue of **A**.

True. A is not invertible $\Leftrightarrow$ dim nul $A \geq 1$

$\Leftrightarrow$ there exists a nonzero vector v such that $Av = 0 = 0v$

zero vec

$\lambda = 0$ scalar

$\Leftrightarrow$ 0 is an eigenvalue.

Problem 7 (continued):

(g) If the square matrix $\mathbf{A}$ is diagonalizable, then the columns of $\mathbf{A}$ are linearly independent.

False. The diagonal matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is certainly diagonalizable, but its columns are not linearly independent. (See the matrix A in Problem #3 for another example.)

(h) If a matrix $\mathbf{U}$ has orthonormal columns, then $\mathbf{U}\mathbf{U}^T = \mathbf{I}$.

False. If U is $n \times k$ where $k < n$, then UU^T is the projection matrix for projection onto the column space of U. The statement would be true if U were an $n \times n$ matrix.