Properties of the Determinant

In order to gain some insight into how we will compute determinants in general, let’s calculate the determinants of all elementary $3 \times 3$ matrices.

**Theorem.** Let $A$ and $B$ be $n \times n$ matrices. Then

1. The matrix $A$ is invertible if and only if $\det A \neq 0$.
2. $\det A^T = \det A$
3. $\det AB = (\det A)(\det B)$
Given the fact that $\det AB = (\det A)(\det B)$, we can consider the determinant of the product $EA$

where $E$ is an elementary matrix.

Row operations and the determinant:

1. Suppose that $B$ is obtained from $A$ by applying exactly one row replacement row operation, then
   \[ \det B = \]

2. Suppose that $B$ is obtained from $A$ by applying exactly one row swap row operation, then
   \[ \det B = \]

3. Suppose that $B$ is obtained from $A$ by applying exactly one row scaling row operation, then
   \[ \det B = \]
Corollary. If $A$ has two identical rows, then
\[ \det A = 0. \]

Proof of the fact that doing a row replacement row operation does not change the determinant: Suppose that
\[
B = \begin{bmatrix}
R_1 \\
\vdots \\
R_i + \alpha R_j \\
\vdots \\
R_n
\end{bmatrix}
\]

where $R_1, R_2, \ldots, R_n$ represent the rows of $A$. 
Example. Consider the $4 \times 4$ matrix

$$
A = \begin{bmatrix}
2 & -2 & 4 & 14 \\
4 & 3 & 1 & 2 \\
-1 & 8 & 6 & 2 \\
2 & -2 & 4 & -3
\end{bmatrix}
$$

Let’s calculate the determinant of $A$ using row operations.
Some practice with the properties of determinants:

Let $A$ and $B$ be $4 \times 4$ matrices with $\det A = 3$ and $\det B = -2$. Compute:

1. $\det AB$

2. $\det B^5$

3. $\det 2A$

4. $\det A^T A$

5. $\det B^{-1} AB$