More on subspaces of vector spaces

**Definition.** A nonempty subset $S$ of a vector space $V$ is a *subspace* of $V$ if

1. the zero vector $0$ is in $S$,

2. (closure under vector addition) for each $v_1$ and $v_2$ in $S$, the vector sum $v_1 + v_2$ is in $S$, and

3. (closure under scalar multiplication) for each $r$ in $\mathbb{R}$ and each $v$ in $S$, the scalar multiple $r v$ is in $S$.

**Note.** A subspace $S$ of a vector space $V$ is a vector space in its own right.

**Example.** Consider the line $x_2 = 3x_1$ in the vector space $\mathbb{R}^2$. 
Example. Consider the line \( x_2 = x_1 + 1 \) in the vector space \( \mathbb{R}^2 \).

Example. Let \( \mathbb{P} \) represent the vector space of all polynomial functions as discussed last class. Is \( \mathbb{P} \) a subspace of the vector space of all functions \( f : \mathbb{R} \to \mathbb{R} \)?
Example. Consider the subset $S = \text{Span}\{x, x^2\}$ within $\mathbb{P}$. Is $S$ a subspace of $\mathbb{P}$?

Theorem. If $v_1, v_2, \ldots, v_p$ are vectors in a vector space $V$, then $\text{Span}\{v_1, v_2, \ldots, v_p\}$ is a subspace of $V$. 
Example. Let $V$ be the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$. Which of the following subsets of $V$ are subspaces of $V$?

1. The set of all constant functions.

2. The set of all functions $f$ such that $f(2) = 1$.

3. The set of all functions $f$ such that $f(2) = 0$.

4. The set of all polynomials of degree 3.

5. The set of all polynomials whose degree is at most 3.

6. The set of all differentiable functions.
Subspaces associated to a matrix

There are three important subspaces associated to an $m \times n$ matrix $A$. Let $c_1, \ldots, c_n$ represent the columns of $A$. That is,

$$A = \begin{bmatrix} c_1 & c_2 & \ldots & c_n \end{bmatrix}.$$ 

These column vectors are vectors in $\mathbb{R}^m$.

Let $r_1, \ldots, r_m$ represent the rows of $A$. That is,

$$A = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{bmatrix}.$$ 

These row vectors are vectors in $\mathbb{R}^n$.

**The column space of $A$.** The column space of $A$ is the span of the columns of $A$. We write

$$\text{Col } A = \text{Span}\{c_1, \ldots, c_n\}.$$ 

**The row space of $A$.** The row space of $A$ is the span of the rows of $A$. We write

$$\text{Row } A = \text{Span}\{r_1, \ldots, r_m\}.$$ 

**The null space of $A$.** The null space of $A$ is the set of all vectors $x$ in $\mathbb{R}^n$ such that

$$Ax = 0.$$ 

The null space of $A$ is denoted by $\text{Nul } A$. 


Theorem. Let \( A \) be an \( m \times n \) matrix. The column space of \( A \) is a subspace of \( \mathbb{R}^m \), and the null space and the row space of \( A \) are subspaces of \( \mathbb{R}^n \).

Application. Any plane through the origin in \( \mathbb{R}^3 \) is a subspace of \( \mathbb{R}^3 \).