The dimension of a vector space

The number of elements in a basis of a vector space is an important quantity associated with the space.

In order to be more precise, we need to distinguish between finite-dimensional vector spaces and infinite-dimensional vector spaces.

**Definition.** A vector space $V$ is finite dimensional if it contains a finite spanning set. Otherwise, $V$ is said to be infinite dimensional.

**Example.** $\mathbb{R}^n$ is spanned by the standard basis $\{e_1, \ldots, e_n\}$. Therefore, it is finite dimensional.

**Example.** The vector space $\mathbb{P}_3$ of all polynomial functions whose degree is at most three is spanned by the basis $\{1, x, x^2, x^3\}$. Therefore, it is finite dimensional.

**Example.** $\mathbb{P}$ is the vector space of all polynomial functions of all degrees. It is infinite-dimensional because it does not contain any finite spanning set. (Why not?)
**Theorem.** Let $V$ be a vector space. Any finite spanning set for $V$ has at least as many elements as any linearly independent subset of $V$.

**Corollary.** Any two bases of a finite-dimensional vector space $V$ have the same number of elements.
Definition. The dimension of a finite-dimensional vector space \( V \) is the number of elements in any basis of \( V \). This nonnegative integer is denoted \( \dim V \).

Examples.

1. \( \dim \mathbb{R}^n = n \)
2. Let \( P \) be the plane \( x_1 + x_2 + x_3 = 0 \) in \( \mathbb{R}^3 \). A basis is

\[
\begin{bmatrix}
1 \\
-1 \\
0
\end{bmatrix}, \begin{bmatrix}
0 \\
1 \\
-1
\end{bmatrix}
\]

Hence, \( \dim P = 2 \).

3. \( \dim \mathbb{P}_3 = 4 \)
4. \( \dim M_{2 \times 3} = 6 \)

Here are a couple of other consequences of the notion of dimension.

Theorem. If \( \dim V = n \), then any set in \( V \) with more than \( n \) vectors must be linearly dependent.

Theorem. If \( H \) is a subspace of \( V \), then \( \dim H \leq \dim V \). In fact, any basis of \( H \) can be expanded to a basis of \( V \).
Suppose that $A$ is an $m \times n$ matrix. How can we determine the dimensions of $\text{Col } A$ and $\text{Nul } A$?

**Example.** Let

\[
A = \begin{bmatrix}
1 & -3 & 4 & -1 & 9 \\
-2 & 6 & -6 & -1 & -10 \\
-3 & 9 & -6 & -6 & -3 \\
3 & -9 & 4 & 9 & 0
\end{bmatrix}.
\]

What relationship is there between the dimensions of $\text{Col } A$ and $\text{Nul } A$?