The span of a set of vectors in $\mathbb{R}^n$

**Definition.** Given vectors $v_1, v_2, \ldots, v_p$ in $\mathbb{R}^n$ and some choice of real numbers $r_1, r_2, \ldots, r_p$, then the vector

$$r_1v_1 + r_2v_2 + \ldots + r_pv_p$$

is said to be a linear combination of the vectors $v_1, v_2, \ldots, v_p$. The numbers $r_1, r_2, \ldots, r_p$ are called the weights of the linear combination.

**Definition.** Suppose that $v_1, v_2, \ldots, v_p$ are vectors in $\mathbb{R}^n$. The set of all possible linear combinations of $v_1, v_2, \ldots, v_p$ is called the

$$\text{Span}\{v_1, v_2, \ldots, v_p\}.$$

Note:

1. Every scalar multiple of each $v_k$ is in $\text{Span}\{v_1, v_2, \ldots, v_p\}$.

2. The zero vector is always in the span of any set of vectors.

3. $\text{Span}\{v_1\}$ is the set of all scalar multiples of $v_1$. 
Example. Let

\[ v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}. \]

What vectors are in \( \text{Span}\{v_1, v_2\} \)?
Example. Consider the vectors

\[ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} -5 \\ -8 \\ 2 \end{bmatrix}. \]

For what values of \( x_3 \) is the vector

\[ \mathbf{b} = \begin{bmatrix} 3 \\ -5 \\ x_3 \end{bmatrix} \]

in \( \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\} \)? What does this result mean geometrically?
The matrix-vector product $Ax$

Let $A$ be an $m \times n$ matrix and $x$ be a vector in $\mathbb{R}^n$. We can define the product $Ax$ as a linear combination of the vectors that come from the columns of $A$.

**Definition.** Let $A$ be an $m \times n$ matrix

$$A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix} = \begin{bmatrix}
A_1 \\
A_2 \\
\vdots \\
A_n
\end{bmatrix},$$

where $A_k$ is the $k$th column of $A$. Given

$$x = \begin{bmatrix}
x_1 \\
\vdots \\
x_n
\end{bmatrix}$$

in $\mathbb{R}^n$, we define the matrix-vector product $Ax$ to be the linear combination

$$x_1A_1 + x_2A_2 + \ldots + x_nA_n.$$

Note that $Ax$ is a vector in $\mathbb{R}^m$.

**Example.**

$$\begin{bmatrix} 3 & -8 \\ -1 & 5 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -4 \\ 2 \end{bmatrix} = -4 \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} -8 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} (-4)(3) + (2)(-8) \\ (-4)(-1) + (2)(5) \\ (-4)(2) + (2)(3) \end{bmatrix} = \begin{bmatrix} -28 \\ 14 \\ -2 \end{bmatrix}$$
Remark. Given an $m \times n$ matrix $A$ and $x \in \mathbb{R}^n$, then the matrix equation

$$Ax = b$$

has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} A_1 & A_2 & \cdots & A_n & b \\ \end{bmatrix} = \begin{bmatrix} A \mid b \end{bmatrix}.$$ 

Theorem. Let $A$ be an $m \times n$ matrix. Then the following three statements are equivalent:

1. For each $b$ in $\mathbb{R}^m$, the equation $Ax = b$ has at least one solution.
2. The columns of $A$ span $\mathbb{R}^m$.
3. The matrix $A$ has a pivot position in every row.

Warning: In this theorem, $A$ is a coefficient matrix. The three statements are not equivalent if $A$ is an augmented matrix.
**Observation.** Note that the $k$th entry in $Ax$ is

$$a_{k1}x_1 + a_{k2}x_2 + \ldots + a_{kn}x_n.$$ 

For example,

$$\begin{bmatrix} * & * \\ 5 & 6 \\ * & * \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} * \\ 5x_1 + 6x_2 \\ * \end{bmatrix}.$$ 

The expression $a_{k1}x_1 + a_{k2}x_2 + \ldots + a_{kn}x_n$ is called the **dot product** of $[a_{k1} \ a_{k2} \ \ldots \ a_{kn}]$ and the vector $x$.

**Theorem.** Let $A$ be an $m \times n$ matrix. Then the matrix-vector product $Ax$ is “linear” in $x$. That is,

1. $A(u + v) = Au + Av$ for all $u$ and $v$ in $\mathbb{R}^n$, and

2. $A(cu) = cAu$ for all $u$ in $\mathbb{R}^n$ and all $c$ in $\mathbb{R}$.