1. (15 points) Row reduce the matrix

\[
A = \begin{bmatrix}
1 & 2 & 1 & 2 \\
-1 & -2 & 0 & -2 \\
2 & 4 & -1 & 6
\end{bmatrix}
\]

to reduced row echelon form (RREF). Do only one row operation at a time and specify that operation when you perform it. Indicate when you first arrive at a matrix in echelon form (REF). What are the pivot positions of \( A \)?

\[
\begin{align*}
A & \sim \begin{bmatrix}
1 & 2 & 1 & 2 \\
0 & 0 & 1 & 0 \\
2 & 4 & -1 & 6
\end{bmatrix} \\
& \sim \begin{bmatrix}
1 & 2 & 1 & 2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 2
\end{bmatrix} \\
& \sim \begin{bmatrix}
1 & 2 & 1 & 2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\end{align*}
\]

\( \text{REF} \)

\[
\begin{align*}
R_1 & \rightarrow R_1 - 2R_3 \\
\end{align*}
\]

\[
\begin{align*}
R_1 & \rightarrow R_1 - R_2 \\
\end{align*}
\]

\( \text{RREF} \)

See above for pivot positions.
2. (16 points) Find the value(s) of $h$ such that the following set of vectors is linearly independent:

\[ \left\{ \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} -5 \\ 7 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ h \end{bmatrix} \right\} \]

\[ A = \begin{bmatrix} 1 & -5 & 1 \\ -1 & 7 & 1 \\ -3 & 8 & h \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 1 \\ 0 & 2 & 2 \\ 0 & -7 & h+3 \end{bmatrix} \]

\[ \sim \begin{bmatrix} 1 & -5 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & h+10 \end{bmatrix} \]

Dependence relations correspond to nontrivial solutions to $Ax = 0$.

If $h \neq -10$, there are no free variables, and the set is linearly independent. If $h = -10$, then there are nontrivial solutions, and the vectors are linearly dependent.
3. (21 points) Which of the following functions \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) are linear? If the function is linear, what is its standard matrix representation? If the function is not linear, justify your answer by giving an example of one of the linearity properties that does not hold.

(a) \( T(x_1, x_2) = (3x_1 - x_2, 2|x_1|) \)

Not linear. For example, \( T(-x_1, -x_2) = (-3x_1 + x_2, 21 - x_1) \)

\[ = (3x_1 + x_2, 2|x_1|) \]

\[ \neq (-1) T(x_1, x_2) \]

(b) \( T(x_1, x_2) = (x_2 - x_1, \sin x_2) \)

Not linear

\( T(0, \pi/2) = (\pi/2, 1) \)

\( T(0, \pi) = (\pi, 0) \neq 2 T(0, \pi/2) \).

(c) \( T(x_1, x_2) = (3x_1 - x_2, 2x_1 + 4x_2) \)

Linear. Let \( A = \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix} \).

Then \( A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 - x_2 \\ 2x_1 + 4x_2 \end{bmatrix} \).
4. (20 points) Given the matrix

\[
A = \begin{bmatrix}
1 & 2 & -7 & 5 \\
0 & 1 & -4 & 0 \\
1 & 0 & 1 & 0 \\
2 & -1 & 6 & 8 \\
\end{bmatrix}
\]

Let \( T : \mathbb{R}^4 \to \mathbb{R}^4 \) be the linear transformation defined by \( T(x) = Ax \). Find all vectors \( x \) in \( \mathbb{R}^4 \) such that

\[
Ax = T(x) = \begin{bmatrix}
8 \\
1 \\
1 \\
9 \\
\end{bmatrix}
\]

Form the augmented matrix \([A | 8 1 9]\).

\[
\begin{bmatrix}
1 & 2 & -7 & 5 & 8 \\
0 & 1 & -4 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 \\
2 & -1 & 6 & 8 & 9 \\
\end{bmatrix} \sim \begin{bmatrix}
1 & 2 & -7 & 5 & 8 \\
0 & 1 & -4 & 0 & 1 \\
0 & -2 & 8 & -5 & -7 \\
0 & -5 & 20 & -2 & -7 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & -7 & 5 & 8 \\
0 & 1 & -4 & 0 & 1 \\
0 & 0 & 0 & -5 & -5 \\
0 & 0 & 0 & -2 & -2 \\
\end{bmatrix} \sim \begin{bmatrix}
1 & 2 & -7 & 5 & 8 \\
0 & 1 & -4 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Consistent system with \( x_3 \) free.
#4 (cont.)

We have \( x_4 = 1 \)

\[
x_2 = 4x_3 + 1
\]

\[
x_1 = -2x_2 + 7x_3 - 5x_4 + 8
\]

\[\Rightarrow x_1 = -2(4x_3 + 1) + 7x_3 - 5 + 8\]

\[\Rightarrow x_1 = -8x_3 + 7x_3 - 2 - 5 + 8\]

\[\Rightarrow x_1 = -x_3 + 1\]

Then

\[
x = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 4 \\ 1 \\ 0 \end{bmatrix}.
\]
5. (28 points) Are the following statements true or false? You will not receive any credit unless you justify your answers. (Note that there are two more parts to this question on the next page.)

(a) Each matrix is row equivalent to a unique matrix in echelon form.

False. The two matrices

\[ A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \]

are row equivalent, and both are in echelon form. It is only the reduced row echelon form that is unique.

(b) An \( m \times n \) matrix \( A \) defines a linear transformation \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) by the formula \( T(x) = Ax \).

False. An \( m \times n \) matrix \( A \) defines a linear transformation

\[ T : \mathbb{R}^n \rightarrow \mathbb{R}^m. \]
Question 5 (continued):

(c) The columns of any $4 \times 5$ matrix are linearly dependent.

True. The columns would be 5 vectors in $\mathbb{R}^4$. Any set of five vectors in $\mathbb{R}^4$ is linearly dependent. The equation $Ax = 0$ would have free variables because $A$ cannot have more than four pivots.

(d) Suppose that $A$ is an $m \times n$ matrix and $B$ is an $n \times p$ matrix. Then the second row of $AB$ is the second row of $A$ multiplied on the right by $B$.

True. The second row multiplied on the right by $B$ is

\[ [a_{21}, a_{22}, \ldots, a_{2n}]B. \]

The $2j$th entry is

\[ a_{21}b_{1j} + a_{22}b_{2j} + \ldots + a_{2n}b_{nj}. \]

By the "dot product" formula for matrix multiplication, this is the same as the $2j$th entry of $AB$. 

\[ \text{M72} \]