1. (10 points) Let

\[ \mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ -7 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}. \]

(a) Write \( \mathbf{v} \) as the sum of a vector in the line spanned by \( \mathbf{w} \) and a vector orthogonal to \( \mathbf{w} \).

\[
\text{proj} \, \mathbf{v} = \left( \frac{\mathbf{v} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}} \right) \mathbf{w} = \frac{6 + 3 - 7}{9 + 1 + 1} = \frac{2}{11} \mathbf{w} = \frac{2}{11} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{6}{11} \\ \frac{2}{11} \\ \frac{2}{11} \end{bmatrix}
\]

Let \( \mathbf{u} = \mathbf{v} - \text{proj} \, \mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ -7 \end{bmatrix} - \begin{bmatrix} \frac{6}{11} \\ \frac{2}{11} \\ \frac{2}{11} \end{bmatrix} = \begin{bmatrix} \frac{2}{11} \\ \frac{7}{11} \\ \frac{5}{11} \end{bmatrix}
\]

\[ \mathbf{v} = \begin{bmatrix} \frac{6}{11} \\ \frac{2}{11} \\ \frac{2}{11} \end{bmatrix} + \begin{bmatrix} \frac{2}{11} \\ \frac{7}{11} \\ \frac{5}{11} \end{bmatrix} \overset{\text{in span} \, \mathbf{w}}{\leadsto} \text{orthogonal to} \, \mathbf{w} \]

(b) Compute the distance of \( \mathbf{v} \) to the line spanned by \( \mathbf{w} \).

\[
\text{distance} = ||\mathbf{u}|| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \frac{1}{11} \sqrt{16^2 + 31^2 + 79^2} = \sqrt{7458} \]

\[ = \frac{\sqrt{7458}}{11} \]
2. (10 points) Let $A$ and $B$ be $4 \times 4$ matrices with $\det A = 2$ and $\det B = -3$. Compute:

(a) $\det 3A = (3)^4 \det A = (81)(2) = 162$

(b) $\det B^3 = (\det B)(\det B)(\det B) = -27$

(c) $\det AB = (\det A)(\det B) = (2)(-3) = -6$

(d) $\det A^T A = (\det A^T)(\det A) = (\det A)(\det A)$
   $\quad = (2)(2) = 4$

(e) $\det B^{-1}AB = (\det B^{-1})(\det A)(\det B)$
   $\quad = \left(\frac{1}{\det B}\right)(\det A)(\det B)$
   $\quad = \det A = 2$. 
3. (14 points) Consider the set $S$ of all vectors

$$
\begin{bmatrix}
  a \\
  b \\
  c \\
  d \\
\end{bmatrix}
$$

in $\mathbb{R}^4$ such that

$$
\begin{align*}
a - 2b + 2c + d &= 0 \\
-3a + 6b - 5c - d &= 0 \\
4a - 8b + 9c + 6d &= 0.
\end{align*}
$$

(a) Why is $S$ a subspace of $\mathbb{R}^4$?

$S = \text{null } A$ where $A = \begin{bmatrix}
  1 & -2 & 2 & 1 \\
  -3 & 6 & -5 & -1 \\
  4 & -8 & 9 & 6
\end{bmatrix}$

(b) Determine the dimension of $S$ and find a basis.

$$
A \sim \begin{bmatrix}
  1 & -2 & 2 & 1 \\
  0 & 0 & 1 & 2 \\
  0 & 0 & 1 & 2
\end{bmatrix} \Rightarrow \begin{bmatrix}
  1 & -2 & 2 & 1 \\
  0 & 0 & 1 & 2
\end{bmatrix}
$$

two free variables ($b$ and $d$) $\Rightarrow \dim S = 2$.

$b = 0$ and $d = 1 \Rightarrow c = -2$
$a = 2b - 2c - d = 3$

$b = 1$ and $d = 0 \Rightarrow c = 0$
$a = 2b - 2c - d = 2$

basis: $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$
4. (14 points) Consider the matrix

\[
A = \begin{bmatrix}
4 & 0 & -2 \\
2 & 5 & 4 \\
0 & 0 & 5
\end{bmatrix}.
\]

(a) Without doing any computation, explain why \( \lambda = 5 \) is an eigenvalue.

From the second column, we see that 
\[AE_2 = 5E_2.\]

(b) What's the "easy" way to show that \( v = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \) is an eigenvector?

\[AV = \begin{bmatrix} -4 \\ 8 \\ 0 \end{bmatrix} = 4v \quad \text{eigenvalue} = 4\]

(c) Find an invertible matrix \( P \) and a diagonal matrix \( D \) such that \( D = P^{-1}AP \). You do not need to calculate \( P^{-1} \).

\[
\text{char pol} = \det(A - \lambda I) = \begin{vmatrix}
4-\lambda & 0 & -2 \\
2 & 5-\lambda & 4 \\
0 & 0 & 5-\lambda
\end{vmatrix}
\]

\[= (5-\lambda) \begin{vmatrix}
4-\lambda & -2 \\
0 & 5-\lambda
\end{vmatrix} \quad \text{\( \lambda = 4, 5 \)}
\]

\[= (5-\lambda)^2(4-\lambda) \]

\[
nul(A - \lambda I) = nul\begin{bmatrix}
-1 & 0 & -2 \\
2 & 0 & 4 \\
0 & 0 & 0
\end{bmatrix} = nul\begin{bmatrix}
-1 & 0 & -2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\( \lambda = 5 \) eigenspace: \( x_1 + 2x_3 = 0 \)
#4 (c) cont.

two free variables:

\[ x_2 = 1 \text{ and } x_3 = 0 \Rightarrow x_1 = 0 \]

eigenvector is \( e_2 \) (already known).

\[ x_2 = 0 \text{ and } x_3 = 1 \Rightarrow x_1 = -2 \]

eigenvector is \( \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \).

matrix \( P \) of eigenvectors

\[ P = \begin{bmatrix} -1 & 0 & -2 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

In this case,

\[ P^4 A P = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \].
5. (14 points) Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformation that first rotates the plane by 45° in the counterclockwise direction, then dilates the plane by a factor of 2, and finally reflects the plane in the $x_1$-axis.

(a) Find the standard matrix representation for $T$.

The left-hand point is $(-\sqrt{2}, -\sqrt{2})$ and the right-hand point is $(\sqrt{2}, -\sqrt{2})$.

The matrix $A$ is:

$$
\begin{bmatrix}
\frac{\sqrt{2}}{\sqrt{2}} & -\frac{\sqrt{2}}{\sqrt{2}} \\
-\frac{\sqrt{2}}{\sqrt{2}} & -\frac{\sqrt{2}}{\sqrt{2}}
\end{bmatrix}
$$

(b) Let $P$ be the parallelogram determined by the two vectors

$$
v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.
$$

Calculate the area of $T(P)$.

$$
\text{area } P = \begin{vmatrix}
1 & 1 \\
1 & 3
\end{vmatrix} = 2
$$

$$
\text{area } T(P) = |\det A| \cdot \text{area}(P) = |-4| \cdot (2) = 8
$$
6. (14 points) Note that there is a second part to this problem on the next page. Recall that a matrix is upper triangular if all of its entries below the diagonal are zero. For example, an upper-triangular $3 \times 3$ matrix $A$ has the form

$$
A = \begin{bmatrix}
  a_{1,1} & a_{1,2} & a_{1,3} \\
  0 & a_{2,2} & a_{2,3} \\
  0 & 0 & a_{3,3}
\end{bmatrix}
$$

where the entries $a_{1,1}$, $a_{1,2}$, $a_{1,3}$, $a_{2,2}$, $a_{2,3}$, and $a_{3,3}$ can be any real numbers.

(a) Show that the subset $S$ of all upper-triangular matrices in $M_{3 \times 3}$ is a vector subspace of $M_{3 \times 3}$.

1. (not necessary) The zero vector is the zero matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. This matrix is upper triangular.

2. Closed under vector addition:
   Given $A$ as above and
   \[ B = \begin{bmatrix}
     b_{11} & b_{12} & b_{13} \\
     0 & b_{22} & b_{23} \\
     0 & 0 & b_{33}
   \end{bmatrix}, \]
   then
   \[ A + B = \begin{bmatrix}
     a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\
     0 & a_{22} + b_{22} & a_{23} + b_{23} \\
     0 & 0 & a_{33} + b_{33}
   \end{bmatrix}, \]
   which is upper triangular.

3. Closed under scalar multiplication:
   Given $A$ as above and $r$ in $\mathbb{R}$
   \[ rA = \begin{bmatrix}
     ra_{11} & ra_{12} & ra_{13} \\
     0 & ra_{22} & ra_{23} \\
     0 & 0 & ra_{33}
   \end{bmatrix}, \]
   which is upper triangular.
Problem 6 (continued):

(b) Specify a basis for $S$ and show that it is a basis. What is the dimension of $S$?

basis consists of six upper triangular matrices:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}.
\]

$\Rightarrow \dim S = 6$.

Need to verify that these six matrices form a basis of $S$.

1) Linearly independent: Given a dependence

\[
r_1 \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} + r_2 \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} + r_3 \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} + r_4 \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
+ r_5 \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix} + r_6 \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
\]

we see that $r_1 = r_2 = r_3 = r_4 = r_5 = r_6 = 0$.

2) Spans $S$. Given $A$ on the previous page,

then

\[
A = a_{11} \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} + a_{12} \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} + a_{13} \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
+ a_{22} \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix} + a_{23} \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix} + a_{33} \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]
7. (24 points) Are the following statements true or false? You must justify your answers to receive any credit.

(a) Row operations on a matrix $A$ can change the linear dependence relations among the rows of $A$.

True. Start with $A_1 = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$
Then row 2 = 2 row 1. Then
$A_1 \sim A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then
row 2 = 0 row 1.

(b) A square matrix is invertible if and only if it is the product of elementary matrices.

True. We know that a matrix $A$ is invertible $\iff$ $A$ is row equivalent to the identity matrix $I$. Each row operation corresponds to multiplication by an elementary matrix, so we have:
$A$ is invertible $\iff E_nE_{n-1}\ldots E_2E_1A = I$.
$\iff A = E_1^{-1}E_2^{-1}\ldots E_{n-1}^{-1}E_n^{-1}$. 
Problem 7 (continued):

(c) A basis is a spanning set that is as large as possible.

False: One basis for $\mathbb{R}^2$ is $\{[1, 0], [1, 2]\}$.

A bigger spanning set is $\{[1, 0], [0, 1], [1, 1]\}$.

In fact, one can always increase the number of vectors in a spanning set, so there is no such thing as a largest spanning set.

(d) If $\lambda$ is an eigenvalue for the $n \times n$ matrix $A$ and $\mu$ is an eigenvalue for the $n \times n$ matrix $B$, then the product $\lambda \mu$ is an eigenvalue for the matrix $AB$.

False. For example, let $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ which has $\lambda = 2, 4$, and let $B = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ which has $\mu = 1, 3$. Then

$AB = \begin{bmatrix} 3 & 3 \\ 1 & 9 \end{bmatrix}$.

$\det(AB - \lambda I) = \begin{vmatrix} 3 - \lambda & 3 \\ 1 & 9 - \lambda \end{vmatrix}$

$= (\lambda - 3)(\lambda - 9) - 3$

$= \lambda^2 - 12\lambda + 24.$

Roots are $\frac{12 \pm \sqrt{144 - 96}}{2}$

$= 6 \pm \sqrt{12}$. 
Problem 7 (continued):

(e) Every projection matrix is orthogonal.

False. Projection onto the line \( x_2 = x_1 \)

is \( P = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \). The columns
are neither orthogonal nor
unit length.

(f) Every projection matrix is diagonalizable.

True. Consider projection onto the
subspace \( W \) of \( \mathbb{R}^n \). If \( \dim W = k \),
then \( W \) is the \( k \) dimensional
eigen space corresponding to \( \lambda = 1 \),
and \( W^\perp \) is the \( n-k \) dimensional
eigen space corresponding to \( \lambda = 0 \).
We can diagonalize a matrix if there
is a basis of \( \mathbb{R}^n \) of eigen vectors. Pick
a basis \( \{u_1, \ldots, u_k\} \) of \( W \) and a basis
\( \{v_1, \ldots, v_{n-k}\} \) of \( W^\perp \). Then
\( \{u_1, \ldots, u_k, v_1, \ldots, v_{n-k}\} \) is the required
basis.