1. (10 points) Let

$$\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ -7 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Write $\mathbf{v}$ as the sum of a vector in the line spanned by $\mathbf{w}$ and a vector orthogonal to $\mathbf{w}$.

$$\text{proj } v = \left(\frac{v \cdot w}{w \cdot w}\right) w \qquad v \cdot w = 6+3-7 = 2$$

$$w \cdot w = 9+1+1 = 11$$

$$= \frac{2}{11}\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{6}{11} \\ \frac{2}{11} \\ \frac{2}{11} \end{bmatrix}$$

Let $u = v - \text{proj } v = \begin{bmatrix} 2 \\ 3 \\ -7 \end{bmatrix} - \begin{bmatrix} \frac{6}{11} \\ \frac{2}{11} \\ \frac{2}{11} \end{bmatrix} = \begin{bmatrix} \frac{16}{11} \\ \frac{31}{11} \\ -\frac{79}{11} \end{bmatrix}$

$$v = \underbrace{\begin{bmatrix} \frac{6}{11} \\ \frac{2}{11} \\ \frac{2}{11} \end{bmatrix}}_{\text{in span } w} + \underbrace{\begin{bmatrix} \frac{16}{11} \\ \frac{31}{11} \\ -\frac{79}{11} \end{bmatrix}}_{\text{orthogonal to } w}$$

(b) Compute the distance of $\mathbf{v}$ to the line spanned by $\mathbf{w}$.

$$\text{distance} = \|u\| = \sqrt{u \cdot u} = \frac{1}{11}\sqrt{16^2 + 31^2 + 79^2}$$

$$= \frac{\sqrt{7458}}{11}$$

2. (10 points) Let $\mathbf{A}$ and $\mathbf{B}$ be $4 \times 4$ matrices with $\det \mathbf{A} = 2$ and $\det \mathbf{B} = -3$. Compute:

(a) $\det 3\mathbf{A}$ $= (3)^4 \det A = (81)(2) = 162$

(b) $\det \mathbf{B}^3$ $= (\det B)(\det B)(\det B) = -27$

(c) $\det \mathbf{AB}$ $= (\det A)(\det B) = (2)(-3) = -6$

(d) $\det \mathbf{A}^{\mathrm{T}}\mathbf{A}$ $= (\det A^T)(\det A) = (\det A)(\det A)$

$= (2)(2) = 4$

(e) $\det \mathbf{B}^{-1}\mathbf{AB}$ $= (\det B^{-1})(\det A)(\det B)$

$= \left(\frac{1}{\det B}\right)(\det A)(\det B)$

$= \det A = 2.$

3. (14 points) Consider the set $S$ of all vectors

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

in $\mathbb{R}^4$ such that

$$\begin{aligned} a - 2b + 2c + d &= 0 \\ -3a + 6b - 5c - d &= 0 \\ 4a - 8b + 9c + 6d &= 0. \end{aligned}$$

(a) Why is $S$ a subspace of $\mathbb{R}^4$?

$S = \text{nul} A$ where $A = \begin{bmatrix} 1 & -2 & 2 & 1 \\ -3 & 6 & -5 & -1 \\ 4 & -8 & 9 & 6 \end{bmatrix}$

(b) Determine the dimension of $S$ and find a basis.

$A \sim \begin{bmatrix} 1 & -2 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

two free variables (b and d) $\Rightarrow \dim S = 2$.

$b = 0$ and $d = 1 \Rightarrow c = -2$
$a = 2b - 2c - d = 3$

$b = 1$ and $d = 0 \Rightarrow c = 0$
$a = 2b - 2c - d = 2$

basis: $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$

4. (14 points) Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}.$$

(a) Without doing any computation, explain why $\lambda = 5$ is an eigenvalue.

From the second column, we see that $Ae_2 = 5e_2$.

(b) What's the "easy" way to show that $\mathbf{v} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ is an eigenvector?

$$A\mathbf{v} = \begin{bmatrix} -4 \\ 8 \\ 0 \end{bmatrix} = 4\mathbf{v} \qquad \text{eigenvalue} = 4$$

(c) Find an invertible matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$. **You do not need to calculate $\mathbf{P}^{-1}$.**

$$\text{char poly} = \det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 0 & -2 \\ 2 & 5-\lambda & 4 \\ 0 & 0 & 5-\lambda \end{vmatrix}$$

$$= (5-\lambda)\begin{vmatrix} 4-\lambda & -2 \\ 0 & 5-\lambda \end{vmatrix} = (5-\lambda)^2(4-\lambda)$$

$$\lambda = 4, 5$$

$$\text{nul}(A - 5I) = \text{nul}\begin{bmatrix} -1 & 0 & -2 \\ 2 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} = \text{nul}\begin{bmatrix} -1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\lambda = 5$ eigenspace : $x_1 + 2x_3 = 0$

#4 (c) cont.

two free variables:

$x_2 = 1$ and $x_3 = 0 \Rightarrow x_1 = 0$

eigenvector is $e_2$ (already known)

$x_2 = 0$ and $x_3 = 1 \Rightarrow x_1 = -2$

eigenvector is $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$.

matrix $P$ of eigenvectors

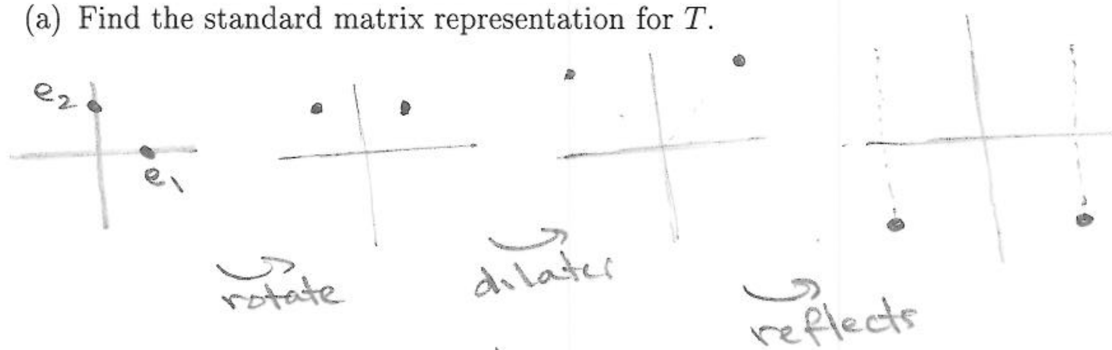
$$P = \begin{bmatrix} -1 & 0 & -2 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In this case,

$$P^{-1}AP = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

5. (14 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that first rotates the plane by $45°$ in the counterclockwise direction, then dilates the plane by a factor of 2, and finally reflects the plane in the $x_1$-axis.

(a) Find the standard matrix representation for $T$.




The left-hand point is $(-\sqrt{2}, -\sqrt{2})$ and the right-hand point is $(\sqrt{2}, -\sqrt{2})$

$$\text{matrix } A = \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & -\sqrt{2} \end{bmatrix}$$

(b) Let $P$ be the parallelogram determined by the two vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Calculate the area of $T(P)$.

$$\text{area } P = \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = 2$$

$$\text{area } T(P) = |\det A| \text{ area}(P)$$
$$= |-4|(2) = 8$$

6. (14 points) **Note that there is a second part to this problem on the next page.** Recall that a matrix is upper triangular if all of its entries below the diagonal are zero. For example, an upper-triangular $3 \times 3$ matrix $\mathbf{A}$ has the form

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ 0 & a_{2,2} & a_{2,3} \\ 0 & 0 & a_{3,3} \end{bmatrix}$$

where the entries $a_{1,1}$, $a_{1,2}$, $a_{1,3}$, $a_{2,2}$, $a_{2,3}$, and $a_{3,3}$ can be any real numbers.

(a) Show that the subset $S$ of all upper-triangular matrices in $M_{3\times3}$ is a vector subspace of $M_{3\times3}$.

(1) (not necessary) The zero vector is the zero matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. This matrix is upper triangular.

(2) closed under vector addition:
Given A as above and

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{bmatrix}.$$

then

$$A+B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ 0 & a_{22}+b_{22} & a_{23}+b_{23} \\ 0 & 0 & a_{33}+b_{33} \end{bmatrix}$$

which is upper triangular.

(3) closed under scalar multiplication:
Given A as above and $r$ in $\mathbb{R}$

$$rA = \begin{bmatrix} ra_{11} & ra_{12} & ra_{13} \\ 0 & ra_{22} & ra_{23} \\ 0 & 0 & ra_{33} \end{bmatrix}$$

which is upper triangular.

Problem 6 (continued):

(b) Specify a basis for $S$ and show that it is a basis. What is the dimension of $S$?

basis consists of six upper triangular matrices:

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}.$$

$\Rightarrow \dim S = 6$.

Need to verify that these six matrices form a basis of $S$.

① linearly independent: Given a dependence relation

$$r_1\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix} + r_2\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix} + r_3\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix} + r_4\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}$$

$$+ r_5\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix} + r_6\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix} = \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix},$$

we see that $r_1 = r_2 = r_3 = r_4 = r_5 = r_6 = 0$.

② spans $S$. Given $A$ on the previous page, then

$$A = a_{11}\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix} + a_{12}\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix} + a_{13}\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}$$

$$+ a_{22}\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix} + a_{23}\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix} + a_{33}\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}.$$

7. (24 points) Are the following statements true or false? You must justify your answers to receive any credit.

   (a) Row operations on a matrix **A** can change the linear dependence relations among the rows of **A**.

True. Start with $A_1 = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$

Then row 2 = 2 row 1. Then

$A_1 \sim A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then

row 2 = 0 row 1.

   (b) A square matrix is invertible if and only if it is the product of elementary matrices.

True. We know that a matrix A is invertible $\Leftrightarrow$ A is row equivalent to the identity matrix I. Each row operation corresponds to multiplication by an elementary matrix, so we have:

A is invertible $\Leftrightarrow E_n E_{n-1} \cdots E_2 E_1 A = I$.

$\Leftrightarrow A = E_1^{-1} E_2^{-1} \cdots E_{n-1}^{-1} E_n^{-1}$.

Problem 7 (continued):

(c) A basis is a spanning set that is as large as possible.

False. One basis for $\mathbb{R}^2$ is $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$.

A bigger spanning set is

$\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$.

In fact, one can always increase the number of vectre in a spanning set. So there is no such thing as a largest spanning set.

(d) If $\lambda$ is an eigenvalue for the $n \times n$ matrix $\mathbf{A}$ and $\mu$ is an eigenvalue for the $n \times n$ matrix $\mathbf{B}$, then the product $\lambda\mu$ is an eigenvalue for the matrix $\mathbf{AB}$.

False. For example, let $A = \begin{bmatrix}3 & 1\\1 & 3\end{bmatrix}$

which has $\lambda = 2, 4$, and let $B = \begin{bmatrix}1 & 0\\0 & 3\end{bmatrix}$

which has $\mu = 1, 3$. Then

$AB = \begin{bmatrix}3 & 3\\1 & 9\end{bmatrix}$. $\det(AB - \lambda I) = \begin{vmatrix}3-\lambda & 3\\1 & 9-\lambda\end{vmatrix}$

$= (\lambda-3)(\lambda-9) - 3 =$

$= \lambda^2 - 12\lambda + 24$.

roots are $\dfrac{12 \pm \sqrt{144-96}}{2}$



$= 6 \pm \sqrt{12}$.

Problem 7 (continued):

(e) Every projection matrix is orthogonal.

False. Projection onto the line $x_2 = x_1$ is $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$. The columns are neither orthogonal nor unit length.

(f) Every projection matrix is diagonalizable.

True. Consider projection onto the subspace $W$ of $\mathbb{R}^n$. If $\dim W = k$, then $W$ is the $k$ dimensional eigenspace corresponding to $\lambda = 1$, and $W^\perp$ is the $n-k$ dimensional eigenspace corresponding to $\lambda = 0$. We can diagonalize a matrix if there is a basis of $\mathbb{R}^n$ of eigenvectors. Pick a basis $\{u_1, \ldots, u_k\}$ of $W$ and a basis $\{v_1, \ldots, v_{n-k}\}$ of $W^\perp$. Then $\{u_1, \ldots, u_k, v_1, \ldots, v_{n-k}\}$ is the required basis