

Bases for vector spaces and subspaces

A basis for V is a spanning set that contains as few vectors as possible.

Definition. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is a *basis* for V if

1. it is linearly independent, and
2. it spans V .

Example. The set $\{x^3, x^2, x, 1\}$ is a basis of \mathbb{P}_3 .

Example. The set $\{x^3, x^3 + x^2, x, 1\}$ is another basis of \mathbb{P}_3 .

We need ways of determining bases of vector spaces and their subspaces. The “casting-out procedure” produces a basis from a spanning set.

The casting-out procedure

Given a vector subspace S spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$, we can obtain a basis B for S by casting out the vectors that are linear combinations of the preceding vectors. More precisely, let

1. $B_1 = \{\mathbf{v}_1\}$ as long as $\mathbf{v}_1 \neq \mathbf{0}$, and
2. for $i \geq 2$,
 - (a) (cast out) $B_i = B_{i-1}$ if \mathbf{v}_i is in $\text{Span } B_{i-1}$, or
 - (b) (keep) $B_i = B_{i-1} \cup \{\mathbf{v}_i\}$ if \mathbf{v}_i is not in $\text{Span } B_{i-1}$.

Then the final result B_k is a basis B for S .

Example. Let's apply the casting-out procedure to the set $\{x^3 + 1, x, x^2, x^2 - x, 4, x^3\}$ of polynomials in \mathbb{P}_3 .

Theorem. (similar to The Spanning Set Theorem, Lay, p. 239) Let $S = \text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$. Then the final result B_k of the casting-out procedure applied to $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a basis for S .

The proof of the casting-out procedure is posted on the web site, and we will not discuss it in class. However, to understand the proof of the theorem, it is helpful to consider the example above along with the sets B_1, B_2, \dots, B_6 . We get

$$B_1 = \{x^3 + 1\}$$

$$B_2 = \{x^3 + 1, x\}$$

$$B_3 = \{x^3 + 1, x, x^2\}$$

$$B_4 = B_3$$

$$B_5 = \{x^3 + 1, x, x^2, 4\}$$

$$B_6 = B_5.$$

Bases for Nul \mathbf{A} and Col \mathbf{A}

Last week we did an example that showed how we can produce a basis for Nul \mathbf{A} .

Example. Find a basis for the column space of

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Example. Find a basis for the column space of

$$\mathbf{B} = \begin{bmatrix} 1 & -2 & 0 & 1 \\ -1 & 2 & 3 & 1 \\ 0 & 0 & -3 & -2 \end{bmatrix}.$$

Fact: Suppose that \mathbf{A} and \mathbf{B} are row equivalent matrices. Then the linear dependence relations among the columns of \mathbf{A} are the same as the linear dependence relations among the columns of \mathbf{B} .

Why?

Warning: If you row reduce a matrix \mathbf{A} to a matrix \mathbf{B} in row echelon form, you identify the pivot columns of \mathbf{A} . To find a basis for $\text{Col } \mathbf{A}$, use the pivot columns of \mathbf{A} . **Do not use the pivot columns of \mathbf{B} .** Row reduction usually changes the column space of a matrix.