Inverses of square matrices

A typical square matrix $\mathbf{A}$ has a multiplicative inverse which we denote by $\mathbf{A}^{-1}$, so

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I} \qquad \text{and} \qquad \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}.$$

However, there are infinitely many square matrices that do not have inverses.

For 2×2 matrices, there is a simple formula for $\mathbf{A}^{-1}$.

Theorem 4. Consider the 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

If $ad - bc \neq 0$, then $\mathbf{A}$ is invertible and

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then $\mathbf{A}$ is not invertible.

Here are some basic properties of inverses.

Theorem 6.

1. If $\mathbf{A}$ is an invertible matrix, then $\mathbf{A}^{-1}$ is invertible and $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.
2. If $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ invertible matrices, then $\mathbf{AB}$ is invertible. Moreover, $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.
3. If $\mathbf{A}$ is an invertible matrix, then $\mathbf{A}^T$ is invertible, and $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$.

Elementary matrices and computing inverses

Definition. An *elementary* matrix is a matrix that is obtained from the identity matrix by applying exactly one elementary row operation.

There are three types of elementary row operations—one for each type of row operation.

What happens to a matrix if we multiply it by an elementary matrix?

Example.

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Algorithm for computing $\mathbf{A}^{-1}$

Form the augmented matrix

$$[\ \mathbf{A} \mid \mathbf{I}\ ].$$

Row reduce this matrix so that the left half becomes the identity matrix. At that point, the right half is $\mathbf{A}^{-1}$.