

A little more on finding a basis for Col \mathbf{A}

On Monday we discussed finding a basis for the column space of a matrix, but I would like to revisit the topic briefly.

Example. The matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

is in echelon form, and $\{\mathbf{A}_1, \mathbf{A}_3\}$ forms a basis of Col \mathbf{A} . Also, note that $\mathbf{A}_1 + \frac{2}{3}\mathbf{A}_3 = \mathbf{A}_4$.

Example. The matrix

$$\mathbf{B} = \begin{bmatrix} 1 & -2 & 0 & 1 \\ -1 & 2 & 3 & 1 \\ 0 & 0 & -3 & -2 \end{bmatrix}.$$

is row equivalent to \mathbf{A} . Note that $\mathbf{B}_1 + \frac{2}{3}\mathbf{B}_3 = \mathbf{B}_4$. Also, Col $\mathbf{A} \neq$ Col \mathbf{B} .

Fact: Suppose that \mathbf{A} and \mathbf{B} are row equivalent matrices. Then the linear dependence relations among the columns of \mathbf{A} are the same as the linear dependence relations among the columns of \mathbf{B} .

Why?

Warning: If you row reduce a matrix \mathbf{B} to a matrix \mathbf{A} in row echelon form, you identify the pivot columns of \mathbf{B} . To find a basis for Col \mathbf{B} , use the pivot columns of \mathbf{B} . **Do not use the pivot columns of \mathbf{A} .** Row operations usually change the column space of a matrix.

Coordinates relative to a basis

A basis for a vector space produces a coordinate system for that space.

Theorem. (Unique Representation Theorem) Let $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for a vector space V . Then every vector \mathbf{v} in V can be represented uniquely as

$$\mathbf{v} = c_1\mathbf{b}_1 + \dots + c_n\mathbf{b}_n.$$

The scalars c_1, \dots, c_n are called the coordinates of \mathbf{v} relative to the basis B .

Example. Consider the spanning set $\{x^3 + 1, x, x^2, x^2 - x, 4, x^3\}$ for the vector space \mathbb{P}_3 . There are infinitely many ways to write a given element of \mathbb{P}_3 as a linear combination of these vectors. For example, consider the polynomial $2x^3 - x^2$. It can be written as

$$(-1)x^2 + 2x^3.$$

It can also be written as $2(x^3 + 1) + (-1)x + (-1)(x^2 - x) + (-\frac{1}{2})(4)$. Because this spanning set is not linearly independent, there are many ways to represent $2x^3 - x^2$ as a linear combination of the vectors.

We produced a basis of \mathbb{P}_3 from this spanning set using the casting-out procedure. The basis is $\{x^3 + 1, x, x^2, 4\}$. What are the coordinates of $2x^3 - x^2$ relative to this basis?

Why are coordinates relative to a given basis unique?

The same vector has different coordinates relative to different bases.

Example. Consider the vector

$$\mathbf{x} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$$

in \mathbb{R}^2 . What are its coordinates relative to the standard basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ and what are its coordinates relative to the basis

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}?$$

Notation. Given the representation $\mathbf{v} = c_1\mathbf{b}_1 + \dots + c_n\mathbf{b}_n$ relative to the basis $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$, then the coordinates can be viewed as a vector in \mathbb{R}^n . This vector is denoted

$$[\mathbf{v}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

Example. What is the coordinate vector for $2x^3 - x^2$ relative to the basis $B = \{x^3 + 1, x, x^2, 4\}$ of \mathbb{P}_3 ?

Change of coordinates matrix

If $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is a basis of \mathbb{R}^n , then the B -coordinates of a vector \mathbf{x} are related to the standard coordinates by the equation

$$\mathbf{x} = c_1 \mathbf{b}_1 + \dots + c_n \mathbf{b}_n.$$

This equation can be rewritten in terms of matrix multiplication as

$$\begin{aligned} \mathbf{x} &= \mathbf{P}_B \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \\ &= \mathbf{P}_B [\mathbf{x}]_B \end{aligned}$$

where \mathbf{P}_B is the matrix

$$\mathbf{P}_B = \left[\begin{array}{c|c|c|c} \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_n \end{array} \right].$$

Since \mathbf{P}_B is invertible, we also have $[\mathbf{x}]_B = (\mathbf{P}_B)^{-1} \mathbf{x}$.

Example. We can double check our computation of the B -coordinates for the vector

$$\mathbf{x} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$$

in \mathbb{R}^2 relative to the basis

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

using these equations.

For any vector space V with basis $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$, the B -coordinates define a nice linear transformation from V onto \mathbb{R}^n . The map is defined by

$$\mathbf{v} \mapsto [\mathbf{v}]_B.$$

Theorem. The coordinate transformation $\mathbf{v} \mapsto [\mathbf{v}]_B$ is a one-to-one linear transformation that maps V onto \mathbb{R}^n .

Definition. A one-to-one linear transformation that maps V onto W is called an isomorphism.

From the vector space point of view, two isomorphic vector spaces have the same structure.

Example. For what n is \mathbb{R}^n isomorphic to \mathbb{P}_3 ?