A little more about coordinate transformations

Last class we saw that the coordinate transformation $[\cdot]_B : \mathbb{P}^3 \to \mathbb{R}^4$ given by the basis $B = \{1, x, x^2, x^3\}$ is an isomorphism between $\mathbb{P}^3$ and $\mathbb{R}^4$.

**Example.** For what $n$ is $\mathbb{R}^n$ isomorphic to $M_{2 \times 3}$?

The dimension of a vector space

The number of elements in a basis of a vector space is an important quantity associated with the space.

In order to be more precise, we need to distinguish between finite-dimensional vector spaces and infinite-dimensional vector spaces.

**Definition.** A vector space $V$ is finite dimensional if it contains a finite spanning set. Otherwise, $V$ is said to be infinite dimensional.

**Example.** The vector space $\mathbb{R}^n$ is spanned by the standard basis $\{e_1, \ldots, e_n\}$. Therefore, it is finite dimensional.

**Example.** The vector space $\mathbb{P}_3$ of all polynomial functions whose degree is at most three is spanned by the basis $\{1, x, x^2, x^3\}$. Therefore, it is finite dimensional.

**Example.** $\mathbb{P}$ is the vector space of all polynomial functions of all degrees. It is infinite-dimensional because it does not contain any finite spanning set. (Why not?)
**Theorem.** Let $V$ be a vector space. Any finite spanning set for $V$ has at least as many elements as any linearly independent subset of $V$.

**Corollary.** Any two bases of a finite-dimensional vector space $V$ have the same number of elements.
Definition. The dimension of a finite-dimensional vector space $V$ is the number of elements in any basis of $V$. This nonnegative integer is denoted $\dim V$.

Examples.

1. $\dim \mathbb{R}^n = n$

2. Let $P$ be the plane $x_1 + x_2 + x_3 = 0$ in $\mathbb{R}^3$. A basis is

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}.$$ 

Hence, $\dim P = 2$.

3. $\dim \mathbb{P}_3 = 4$

4. $\dim M_{2 \times 3} = 6$

Here are a couple of other consequences of the notion of dimension.

Theorem. If $\dim V = n$, then any set in $V$ with more than $n$ vectors must be linearly dependent.

Theorem. If $H$ is a subspace of $V$, then $\dim H \leq \dim V$. In fact, any basis of $H$ can be expanded to a basis of $V$. 
Suppose that $A$ is an $m \times n$ matrix. How can we determine the dimensions of $\text{Col } A$ and $\text{Nul } A$?

**Example.** Let

$$A = \begin{bmatrix}
1 & -3 & 4 & -1 & 9 \\
-2 & 6 & -6 & -1 & -10 \\
-3 & 9 & -6 & -6 & -3 \\
3 & -9 & 4 & 9 & 0
\end{bmatrix}. $$

What relationship is there between the dimensions of $\text{Col } A$ and $\text{Nul } A$?