How do we find the eigenvalues?

**Note:** The number \( \lambda \) is an eigenvalue for the matrix \( A \) if and only if the homogeneous system

\[
(A - \lambda I) x = 0
\]

has a nontrivial solution.

By the Invertible Matrix Theorem,

\[
(A - \lambda I) x = 0
\]

has a nontrivial solution if and only if the matrix \( (A - \lambda I) \) is not invertible.

The number \( \lambda \) is an eigenvalue for the matrix \( A \) if and only if

\[
\det(A - \lambda I) = 0.
\]

**Example.** Let

\[
A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.
\]
**Theorem.** If a matrix is upper/lower triangular, then its eigenvalues are the entries along the main diagonal.

Now I want to use the computer to examine the characteristic polynomial for various matrices.

**Example.** Let

\[
A = \begin{bmatrix}
3 & -2 \\
-1 & 0
\end{bmatrix}.
\]
Example. Let

$$A = \begin{bmatrix}
2 & 1 & -1 \\
1 & 2 & -1 \\
1 & 1 & 0
\end{bmatrix}.$$
In theory, any polynomial $p(\lambda)$ can be factored into irreducible linear and quadratic factors using real numbers. For example, consider the polynomial

$$p(\lambda) = \lambda^9 + 8\lambda^8 + 36\lambda^7 + 94\lambda^6 + 143\lambda^5 + 98\lambda^4 - 48\lambda^3 - 160\lambda^2 - 132\lambda - 40.$$ 

This polynomial factors into

$$p(\lambda) = (\lambda^2 + 2\lambda + 2)^2(\lambda^2 + 3\lambda + 10)(\lambda + 1)^2(\lambda - 1).$$

The **algebraic multiplicity** of an eigenvalue $\lambda_0$ is the number of times that the factor $(\lambda - \lambda_0)$ appears in the factorization of the characteristic polynomial $p(\lambda)$. The **geometric multiplicity** of $\lambda_0$ is the dimension of its eigenspace.

**Theorem.** The geometric multiplicity of an eigenvalue is always less than or equal to its algebraic multiplicity.

**Example.** Consider the matrices

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$