The vector space $\mathbb{R}^n$

**Definition.** The vector space $\mathbb{R}^n$ is the set of all $n$-tuples of real numbers. That is, $\mathbb{R}^n$ is the set of all possible $n \times 1$ “column vectors” of the form

\[
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix},
\]

where $x_k$ is a real number for $k = 1, 2, \ldots, n$.

Vector addition: Given two vectors

\[
v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix},
\]

the vector sum $v + w$ is the vector

\[
\begin{bmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{bmatrix}.
\]

Vector addition can be visualized using the parallelogram rule.

Scalar multiplication: Given a vector

\[
v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}
\]

and a real number (a “scalar”) $r$, then

\[
rv = \begin{bmatrix} rv_1 \\ \vdots \\ rv_n \end{bmatrix}.
\]
Algebraic Properties of $\mathbb{R}^n$

For all $u, v, w$ in $\mathbb{R}^n$ and all scalars $c$ and $d$:

- $u + v = v + u$ \hspace{1cm} \text{commutative property}
- $(u + v) + w = u + (v + w)$ \hspace{1cm} \text{associative property}
- $u + 0 = 0 + u = u$ \hspace{1cm} \text{zero vector}
- $u + (-u) = -u + u = 0$ \hspace{1cm} $-u$ denotes $(-1)u$
- $c(u + v) = cu + cv$ \hspace{1cm} \text{distributive property}
- $(c + d)u + cu + du$
- $c(du) = (cd)u$
- $1u = u$

**Example.** The set of all points $(x_1, x_2, x_3)$ in $\mathbb{R}^3$ that satisfy the equation

$$x_1 + x_2 + x_3 = 0$$

is a plane. How can we describe this plane using vector operations?

**Definition.** Given vectors $v_1, v_2, \ldots, v_p$ in $\mathbb{R}^n$ and some choice of real numbers $r_1, r_2, \ldots, r_p$, then the vector

$$r_1v_1 + r_2v_2 + \ldots + r_pv_p$$

is said to be a linear combination of the vectors $v_1, v_2, \ldots, v_p$. The numbers $r_1, r_2, \ldots, r_p$ are called the weights of the linear combination.

**Examples.**
Important question: Given vectors \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p \) as well as a vector \( \mathbf{b} \), is \( \mathbf{b} \) a linear combination of \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p \)?

**Example.** Given

\[
\begin{align*}
\mathbf{v}_1 &= \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, & \mathbf{v}_2 &= \begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix}, & \mathbf{v}_3 &= \begin{bmatrix} 2 \\ 3 \\ 3 \\ 1 \end{bmatrix}, & \mathbf{v}_4 &= \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}
\end{align*}
\]

and

\[
\begin{align*}
\mathbf{b}_1 &= \begin{bmatrix} -3 \\ -2 \\ 3 \\ -1 \end{bmatrix}, & \mathbf{b}_2 &= \begin{bmatrix} 5 \\ 6 \\ 1 \\ 1 \end{bmatrix}.
\end{align*}
\]

Is either \( \mathbf{b}_1 \) or \( \mathbf{b}_2 \) a linear combination of \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \)?

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Definition. Suppose that $v_1, v_2, \ldots, v_p$ are vectors in $\mathbb{R}^n$. The set of all possible linear combinations of $v_1, v_2, \ldots, v_p$ is called the

$$\text{span}\{v_1, v_2, \ldots, v_p\}.$$ 

Note:

1. Every scalar multiple of each $v_k$ is in $\text{span}\{v_1, v_2, \ldots, v_p\}$.

2. The zero vector is always in the span of any set of vectors.

3. The $\text{span}\{v_1\}$ is the set of all scalar multiples of $v_1$. 
